THE RIEMANNIAN L^2 TOPOLOGY ON THE MANIFOLD OF RIEMANNIAN METRICS

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Abstract. We study the manifold of all Riemannian metrics over a closed, finite-dimensional manifold. In particular, we investigate the topology on the manifold of metrics induced by the distance function of the L^2 Riemannian metric—so called because it induces an L^2 topology on each tangent space. It turns out that this topology on the tangent spaces gives rise to an L^1-type topology on the manifold of metrics itself. We study this new topology and its completion, which agrees homeomorphically with the completion of the L^2 metric. We also give a user-friendly criterion for convergence (with respect to the L^2 metric) in the manifold of metrics.

1. Introduction

Let M be a smooth, closed, oriented, finite-dimensional manifold, and let $\mathcal{M}$ denote the space of all Riemannian metrics on M. The space $\mathcal{M}$ is naturally a Fréchet manifold, and it possesses a canonical L^2 Riemannian metric. (So called because it induces an L^2-type scalar product on the *tangent spaces* of $\mathcal{M}$.) Despite the fact that $\mathcal{M}$ is a contractible space, the L^2 metric has rich local geometry—for instance, its curvature is nonnegative, and its geodesics are explicitly computable [7, 8]. The L^2 metric has arisen in Teichmüller theory [13], as well as in studies of the moduli space of Riemannian metrics [6].

In [3] and [5], we made steps towards understanding the global geometry of the L^2 metric. In particular, we showed that the L^2 Riemannian metric induces a metric space structure on $\mathcal{M}$. (See Section 2.1 for a discussion of why this result is nontrivial.) We also gave a natural identification of the completion of $\mathcal{M}$—with respect to the L^2 metric—as a quotient space of the space of all measurable, finite-volume Riemannian semimetrics on M (Theorem 2.6).

In this paper, we carry this study one step further by giving a simplified description of the topology induced by the distance function d of the L^2 metric on $\mathcal{M}$, as well as on the completion $\overline{\mathcal{M}}$. Interestingly, the L^2 topology on the tangent spaces of $\mathcal{M}$ translates—via d—into an L^1-type topology on $\mathcal{M}$ and $\overline{\mathcal{M}}$. An analogous result can be seen in [4, Thm. 5.2], where the L^2 metric induces an $L^{n/2}$-type topology on the space of metrics conformally equivalent to a given metric (where n denotes the dimension of M.)

We can describe the main result of the paper as follows. Let $\mathcal{S}_x$ denote the set of symmetric $(0,2)$-tensors based at $x \in M$, and let $\mathcal{M}_x \subset \mathcal{S}_x$ be those tensors that induce a positive-definite scalar product on T_xM. (Thus, $\mathcal{M}$ is given by the smooth sections of the bundle $\cup_{x \in M} \mathcal{M}_x$.) We define a quotient space by

$$\overline{\mathcal{M}_x} := \mathrm{cl}(\mathcal{M}_x) \big/ \partial \mathcal{M}_x,$$

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where $\mathrm{cl}(\mathcal{M}_x)$ denotes the closure of $\mathcal{M}_x \subset \mathcal{S}_x$ and $\partial\mathcal{M}_x$ denotes the boundary of $\mathcal{M}_x$. In other words, $\overline{\mathcal{M}_x}$ is given by positive *semi*-definite tensors at x, where we identify all tensors that are not positive definite. We will see later (Theorem 2.6) that in a precise sense, the completion $\overline{(\mathcal{M}, d)}$ is given by the measurable sections of the bundle $\cup_{x\in M}\overline{\mathcal{M}_x}$ that have finite total volume.

Let $g \in \mathcal{M}$ be an arbitrary reference metric. Then we have the following result.

Theorem (Theorem 4.13)**.** *For each $x \in M$, there exists a metric (in the sense of metric spaces) θ^g_x on $\overline{\mathcal{M}_x}$ such that the topology of $\overline{(\mathcal{M}, d)}$ agrees with the L^1 topology of θ^g_x. That is, for $g_0, g_1 \in \overline{(\mathcal{M}, d)}$, let*

$$\Theta_M(g_0, g_1) := \int_M \theta^g_x(g_0(x), g_1(x))\, d\mu_g,$$

where μ_g denotes the volume form induced by g. Then the topology induced by the metric Θ_M agrees with the topology of $\overline{(\mathcal{M}, d)}$, and $\overline{(\mathcal{M}, d)}$ is complete with respect to Θ_M.

When studying the topology of d, it is desirable to swap this Riemannian distance function for the simpler description of the above theorem in terms of the L^1 topology of a bundle of metric spaces over M. (See Sections 2.1 and 3.1.) In particular, calculating or estimating Θ_M involves first computing with θ^g_x (a finite-dimensional problem) for each x, and then integrating the results over M. On the other hand, calculating or estimating d involves considering infima of lengths of paths in $\mathcal{M}$ with respect to the L^2 Riemannian metric, a decidedly infinite-dimensional problem. We will give some examples of the utility of this approach in Section 4.3, where we show the discontinuity of numerous geometric quantities on M with respect to d.

The eventual goal of this effort is an understanding of the structure induced by d on the moduli space of Riemannian metrics (sometimes also called *superspace*). This is the quotient space $\mathcal{M}/\mathcal{D}$, where $\mathcal{D}$ denotes the diffeomorphism group of M, acting on $\mathcal{M}$ by pull-back. Since $\mathcal{D}$ acts by isometries on $\mathcal{M}$ with the L^2 metric [4, §6.1.2], d induces a *pseudo*metric-space structure on $\mathcal{M}/\mathcal{D}$ (which not a manifold, but rather a stratified space [2]). It would be interesting to see what our results can say about the completion and metric geometry of $\mathcal{M}/\mathcal{D}$, but the first question one must ask is whether $\mathcal{M}/\mathcal{D}$ is a metric space with this pseudometric. This seems to be a difficult question—see the discussion and examples following Theorem 4.15 for more on this. Nevertheless, we hope that the theorem quoted above may give us some more useful tools for studying these issues in future papers.

This paper is organized as follows. In Section 2, we review the definitions and previous results that we will require. This includes a discussion of the fundamentals on the manifold of metrics and the L^2 metric, the completion of $\mathcal{M}$, and some structures and properties that were laid out in our previous works [3] and [5]. We also include a few novel results and extensions of previous results that will be useful to us in subsequent sections.

In Section 3, we include a detailed discussion of the metric Θ_M given in the above theorem. In particular, we will examine the relationship between Θ_M and volume, as well as describing the complete L^1 space determined by θ^g_x on the bundle $\cup_{x\in M}\overline{\mathcal{M}_x}$.

Finally, in Section 4, we give the proof of the above-quoted main theorem. In addition, at the end of the section we give an alternative characterization of convergence with respect to d when the limit is an element of $\mathcal{M}$ (as opposed to the completion). This in fact gives a relatively easy to verify criterion for convergence—it is simply a kind of convergence in measure, together with a strong convergence of the volume forms.

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2. The manifold of Riemannian metrics

In this section, we review some of the structures and results relating to the geometry of the L^2 metric's distance function that were introduced in [3] and [5]. We will draw on these results throughout the rest of the paper. In addition, we introduce a few new concepts and some new notation that will be important for us. Most of the facts stated here can be found in greater detail in [3, §2] and [5, §2]. (An even more elementary discussion can be found in [4, Ch. 2].)

To begin, though, we must recall some fundamentals on the manifold of Riemannian metrics and the L^2 metric.

2.1. The manifold of metrics.

The facts stated in this section were established in [6], [7] and [8], to which we refer for further details. A detailed overview is also given in [4, Ch. 2].

Let M be a C^∞-smooth, closed, oriented manifold of dimension n. We denote by $\mathcal{S}$ the vector space of smooth, symmetric $(0,2)$-tensor fields on M. It is a Fréchet space when equipped with the family of Sobolev H^s norms for $s \in \mathbb{N}$ [4, §2.5.1].

The set $\mathcal{M}$ of all smooth Riemannian metrics on M is an open, positive cone in $\mathcal{S}$. As such, $\mathcal{M}$ is trivially a Fréchet manifold. Additionally, its tangent space $T_g\mathcal{M}$ at any point $g \in \mathcal{M}$ can be canonically identified with $\mathcal{S}$.

For each $x \in \mathcal{M}$, we define $\mathcal{S}_x := S^2 T^*_x M$ to be the set of all symmetric $(0,2)$-tensors based at x. We denote by $\mathcal{M}_x \subset \mathcal{S}_x$ the open, positive cone of all such tensors that induce a positive definite scalar product on $T_x M$. For each $a \in \mathcal{M}_x$, there is a natural scalar product on $\mathcal{S}_x$ given by

$$\langle b, c\rangle_a := \operatorname{tr}_a(bc) = a^{ij} a^{lm} b_{il} c_{jm} \quad \text{for all } b, c \in \mathcal{S}_x. \tag{2.1}$$

The last expression in the above requires the choice of some coordinates around x, but the resulting value will clearly be coordinate-independent. Furthermore, $\langle \cdot, \cdot \rangle_a$ is positive definite for each $a \in \mathcal{M}_x$ [4, Lemma 2.35]. We will denote the norm associated to (2.1) by $|\cdot|_a$, i.e.,

$$|b|_a = \sqrt{\langle b, b\rangle_a}. \tag{2.2}$$

By integrating the scalar product (2.1), we can obtain a scalar product on elements of $\mathcal{S}$, giving us a Riemannian metric on $\mathcal{M}$. This is the L^2 *metric*, and explicitly, it is given by the following: For $g \in \mathcal{M}$ and $h, k \in \mathcal{S} \cong T_g\mathcal{M}$,

$$(h, k)_g := \int_M \operatorname{tr}_g(hk)\, d\mu_g, \tag{2.3}$$

where μ_g denotes the volume form induced by g. This is indeed a Riemannian metric. Firstly, it is positive definite, as $\langle \cdot, \cdot \rangle_{g(x)}$ is for each $x \in M$. And secondly, $(\cdot, \cdot)$ varies smoothly with g, as shown by Ebin [6, pp. 18–19]. Additionally, this metric is invariant under the diffeomorphism group $\mathcal{D}$, which acts by pull-back [4, §6.1.2]. (That is, $\mathcal{D}$ acts by isometries.) Throughout the rest of this paper, we denote the Riemannian distance function

of $(\cdot,\cdot)$ by d. We denote the norm on $\mathcal{S}$ induced by (2.3) by $\|\cdot\|_g$, that is,

$$\|h\|_g = \sqrt{(h,h)_g}.$$

The curvature (cf. [7, §1], [8, §2.5–2.9]) and geodesics (cf. [7, Thm. 2.3], [8, Thm. 3.2]) of the L^2 metric have been explicitly computed. We will not need them here, except for some very special geodesics. If we let $\mathcal{P} \subset C^\infty(M)$ denote the space of smooth, positive functions on M, then $\mathcal{P}$ acts on $\mathcal{M}$ by pointwise multiplication (conformal changes), and we have the following result.

Proposition 2.1 ([7, Prop. 2.1])**.** *The geodesic starting at $g_0 \in \mathcal{M}$ with initial tangent vector ρg_0, where $\rho \in C^\infty(M)$, is given by*

$$g_t = \left(1 + n\frac{t}{4}\rho\right)^{4/n} g_0.$$

In particular, the exponential mapping $\exp_{g_0}$ is a diffeomorphism from the open set of functions $\{\rho \in \mathcal{P} \mid \rho > -4/n\}$ onto $\mathcal{P} \cdot g_0$.

We must remark here that the L^2 metric is a so-called *weak Riemannian metric* (cf. [5, §3]), which means that each tangent space $T_g\mathcal{M}$ is incomplete with respect to $(\cdot,\cdot)_g$, or equivalently that the topology induced by $(\cdot,\cdot)_g$ on $T_g\mathcal{M}$ is weaker than the manifold topology. In fact, as a consequence, standard results in Riemannian geometry—even the existence of the Levi-Civita connection, curvature tensor, and geodesics—do not hold *a priori*. However, Ebin [6, §4] gave a direct proof that the Levi-Civita connection of the L^2 metric exists, and so in particular, the curvature tensor and geodesics exist as well. One can even show [8, Thm. 3.4] that the exponential mapping $\exp_g$ at $g \in \mathcal{M}$ is a real-analytic diffeomorphism between subsets of $T_g\mathcal{M}$ and $\mathcal{M}$ that are open in the manifold topology.

On the other hand, a serious difficulty in studying the distance function d is that $\exp_g$ is *not* defined on any subset of $T_g\mathcal{M}$ that is open with respect to $(\cdot,\cdot)_g$, and its image does not contain any open d-ball around g. In such a situation, it can happen that the Riemannian distance function is only a *pseudo*metric, i.e., positive definiteness is not guaranteed. (See [10, 11] for examples where the distance function even vanishes everywhere!) As we showed in [5], though, this is not the case here—d in fact induces a metric space structure on $\mathcal{M}$. Nevertheless, there do exist points that are arbitrarily close with respect to d, yet are not connected by a geodesic. This is another reason to pursue our alternative way to estimate d.

2.2. **The completion of $\mathcal{M}$.**

Convention 2.2. For the remainder of the paper, we fix an element $g \in \mathcal{M}$. Whenever we refer to the L^2 norm $\|\cdot\|_g$ and L^2-convergence, we mean that induced by g unless we explicitly state otherwise. The designation nullset refers to Lebesgue-measurable subsets of M that have measure zero with respect to the volume form μ_g of g. If we say that something holds almost everywhere, we mean that it holds outside of a μ_g-nullset.

If we have a tensor $h \in \mathcal{S}$, we denote by the capital letter H the tensor obtained by raising an index with g, i.e., locally $H^i_j := g^{ik}h_{kj}$. We sometimes also write $H = g^{-1}h$. Given a point $x \in M$ and an element $a \in \mathcal{M}_x$, the capital letter A means the same—i.e., we assume some coordinates and write $A = g(x)^{-1}a$, though for readability we will generally omit x from the notation.

To give the description of the completion of $\mathcal{M}$ mentioned in the introduction, we will have to consider generalizations of Riemannian metrics. In particular, we must allow degenerations in both regularity and positive definiteness. The next definition covers these objects.

Definition 2.3. We denote by S^2T^*M the bundle of symmetric $(0,2)$ tensors on M. A section of S^2T^*M (a symmetric $(0,2)$-tensor field) that induces a positive semi-definite scalar product on each tangent space of M is called a *Riemannian semimetric* (or simply *semimetric*). (Note we do not make any assumptions on the regularity of this section.)

We call a semimetric $\tilde{g}$ *measurable* if it is a measurable section of S^2T^*M, and we denote by $\mathcal{M}_m$ the space of all measurable semimetrics on $\mathcal{M}$.

Note that if $\tilde{g} \in \mathcal{M}_m$, then $\mu_{\tilde{g}} := \sqrt{\det \tilde{g}}\, dx^1 \wedge \cdots \wedge dx^n$ is a measurable n-form on M with nonnegative coefficient in each coordinate chart. Thus, it induces a (Lebesgue) measure on M in the usual way. This measure is absolutely continuous with respect to our standard measure μ_g. In particular, a sequence that converges a.e. with respect to μ_g converges a.e. with respect to $\mu_{\tilde{g}}$.

For any measurable subset $E \subseteq M$, we denote by $\mathrm{Vol}(E, \tilde{g})$ (sometimes also denoted $\mu_{\tilde{g}}(E)$) the measure of the subset E with respect to $\mu_{\tilde{g}}$. Furthermore, we define

$$\mathcal{M}_f := \{\tilde{g} \in \mathcal{M}_m \mid \mathrm{Vol}(M, \tilde{g}) < \infty\},$$

so that $\mathcal{M}_f$ is the space of *finite-volume measurable semimetrics.*

Note that if ν is a measurable, positive n-form (meaning with positive coefficient) and μ is a measurable, nonnegative n-form, then there exists a unique nonnegative function (μ/ν) on M that satisfies

$$\mu = \left(\frac{\mu}{\nu}\right)\nu.$$

In particular, if $\nu(M) < \infty$ (the only case we will be concerned with here), then (μ/ν) is the Radon-Nikodym derivative of μ with respect to ν.

The nonnegative n-form of a semimetric $\tilde{g}$ vanishes at exactly those points where $\tilde{g}$ fails to be positive definite. This motivates the following definition.

Definition 2.4. Let $\tilde{g} \in \mathcal{M}_m$. The *deflated set* of $\tilde{g}$ is defined by

$$X_{\tilde{g}} := \{x \in M \mid \det \tilde{G}(x) = 0\}.$$

Analogously, if $\{g_k\} \in \mathcal{M}_m$ is a sequence, then we define

$$D_{\{g_k\}} := \{x \in M \mid \det G_k(x) \to 0\}.$$

Note that, clearly, $\mathrm{Vol}(X_{\tilde{g}}, \tilde{g}) = 0$ for all $\tilde{g} \in \mathcal{M}_m$.

We next define an equivalence relation on $\mathcal{M}_m$ by saying $g_0 \sim g_1$ if and only if the following two statements hold:

(1) The degenerate sets X_{g_0} and X_{g_1} agree up to a nullset; and
(2) $g_0(x) = g_1(x)$ for almost every $x \in M \setminus (X_{g_0} \cup X_{g_1})$.

In other words, we say $g_0 \sim g_1$ if and only if $g_0(x)$ and $g_1(x)$ differ only where they are both deflated (up to a nullset). Denote by $\widehat{\mathcal{M}_m} := \mathcal{M}_m/\sim$ and $\widehat{\mathcal{M}_f} := \mathcal{M}_f/\sim$ the quotients by this equivalence relation.

There is a natural notion of convergence that allows us to give an element of $\widehat{\mathcal{M}_f}$ as the limit of certain sequences in $\mathcal{M}$. This is defined as follows.

Definition 2.5. Let $\{g_k\}$ be a sequence in $\mathcal{M}$, and let $[g_0] \in \widehat{\mathcal{M}_f}$. We say that $\{g_k\}$ *ω-converges* to $[g_0]$ if for every representative $g_0 \in [g_0]$, the following holds:

(1) $\{g_k\}$ is d-Cauchy,
(2) X_{g_0} and $D_{\{g_k\}}$ differ at most by a nullset,
(3) $g_k(x) \to g_0(x)$ for a.e. $x \in M \setminus D_{\{g_k\}}$, and
(4) $\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty$.

We call $[g_0]$ the *ω-limit* of the sequence $\{g_k\}$ and write $g_k \xrightarrow{\omega} [g_0]$.

More generally, if $\{g_k\}$ is a d-Cauchy sequence containing a subsequence that ω-converges to $[g_0]$, then we say that $\{g_k\}$ *ω-subconverges* to $[g_0]$.

Note that the definition of ω-convergence requires a sequence to be d-Cauchy—this is not guaranteed by conditions 2 and 3. Note also that conditions 2 and 3 are the ones that give the substantial properties of an ω-convergent sequence. Condition 4 is merely technical, and can always be achieved by passing to a subsequence, provided the sequence is d-Cauchy.

It is not hard to see [3, Lemma 4.5] that if one representative $g_0 \in [g_0]$ satisfies the conditions in the above definition, then all representatives do. Therefore, for convenience we will usually just write things like $g_k \xrightarrow{\omega} g_0$, even when the equivalence class of g_0 is meant.

Finally, we recall the basic facts about completions of metric spaces, as well as fix notation. As with any metric space, the completion of $(\mathcal{M}, d)$ is a quotient space of the set of Cauchy sequences in $\mathcal{M}$. We define a pseudometric, which for simplicity is also denoted by d, on Cauchy sequences in $\mathcal{M}$ by

$$d(\{g_k\}, \{\tilde{g}_k\}) = \lim_{k \to \infty} d(g_k, \tilde{g}_k).$$

That this limit exists is a straightforward argument using the Cauchy sequence property. It is not hard to see that this d is only a pseudometric on the set of Cauchy sequences, so to get a metric space, we must identify Cauchy sequences with distance zero in this pseudometric. Thus, we write $\{g_k\} \sim \{\tilde{g}_k\}$ if and only if $\lim d(g_k, \tilde{g}_k) = 0$, and define

$$\overline{\mathcal{M}} = \{\text{Cauchy sequences } \{g_k\} \subset \mathcal{M}\} \big/ \sim.$$

Note that $\mathcal{M}$ is isometrically embedded in $\overline{\mathcal{M}}$ by mapping a point $g \in \mathcal{M}$ to the constant sequence $\{g\}$. Furthermore, if $\{g_k\}$ is a Cauchy sequence, then any subsequence $\{g_{k_l}\}$ is equivalent to the original sequence. Therefore, we may pass to subsequences as we like and still be talking about the same element of $\overline{\mathcal{M}}$.

By the results of [3] (Theorems 4.17, 4.27, 4.39, and 5.14, as well as Corollary 4.21), each Cauchy sequence in $\mathcal{M}$ ω-subconverges to some limit $[g_0] \in \widehat{\mathcal{M}_f}$. Furthermore, two Cauchy sequences ω-subconverge to the same limit if and only if they are equivalent (i.e., represent the same element of $\overline{\mathcal{M}}$). And finally, for each element $[g_0] \in \widehat{\mathcal{M}_f}$, there exists some Cauchy sequence in $\mathcal{M}$ ω-subconverging to it. Putting this together, we have the following theorem.

Theorem 2.6 ([3, Thm. 5.17])**.** *There exists a natural bijection $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$. The map Ω assigns to each equivalence class of Cauchy sequences in $\mathcal{M}$ the unique element of $\widehat{\mathcal{M}_f}$ that each of its members ω-subconverge to.*

One goal of this paper is to replace ω-convergence with a clearer notion. A slightly unsatisfactory element of the map Ω is that it requires passing to subsequences of the original Cauchy sequence. This is not much of a problem, since subsequences are equivalent to

the original sequence. The bigger problem is that ω-convergence does not identify Cauchy sequences—it assumes a sequence is Cauchy, and the theorems of [3] essentially tell us that by passing to a subsequence, we obtain the other conditions of ω-convergence. By the end of this paper, however, we will have a more or less explicit condition for a sequence in $\mathcal{M}$ to be Cauchy, as well as a more complete understanding of how it converges to a limit in $\widehat{\mathcal{M}_f}$.

In light of the above results, we will denote the metric that Ω induces on $\widehat{\mathcal{M}_f}$ again by d, and do the same for the pseudometric thus induced on $\mathcal{M}_f$. So if we write $d(g_0, g_1)$ with $g_0, g_1 \in \mathcal{M}_f$, it is understood that this is the same as $d(\{g_0^k\}, \{g_1^k\}) = \lim d(g_0^k, g_1^k)$, where $\{g_0^k\}$ and $\{g_1^k\}$ are sequences ω-converging to g_0 and g_1, respectively.

2.3. **(Quasi-)Amenable subsets.** We will need uniform notions of a Riemannian metric on the base manifold M being "not too large" and "not too small". To do so, we must first fix a "good" coordinate chart on M in which we can evaluate the coefficients of a metric.

Definition 2.7. We call a finite atlas of coordinates $\{(U_\alpha, \phi_\alpha)\}$ for M *amenable* if for each U_α, there exist a compact set K_α and a different coordinate chart (V_α, ψ_α) (which does not necessarily belong to $\{(U_\alpha, \phi_\alpha)\}$) such that

$$U_\alpha \subset K_\alpha \subset V_\alpha \quad \text{and} \quad \phi_\alpha = \psi_\alpha | U_\alpha.$$

Remark 2.8. One nice property of an amenable atlas is the following. Since each chart of an amenable atlas is a relatively compact subset of a different chart on M, we see that given any metric $g \in \mathcal{M}$, the coefficients g_{ij} of the metric are bounded functions in each chart.

Convention 2.9. For the remainder of the paper, we work over a fixed amenable coordinate atlas $\{(U_\alpha, \phi_\alpha)\}$ for all computations and concepts that require local coordinates. If we say that a statement in local coordinates holds at each $x \in M$, then it is understood that the statement should hold in each coordinate chart of $\{(U_\alpha, \phi_\alpha)\}$ containing x.

With this convention, we can say what it means for a Riemannian semimetric to be "not too large".

Definition 2.10. We call a Riemannian semimetric *bounded* if we can find a constant C such that for all $x \in M$ and all $1 \leq i, j \leq n$, we have $|g_{ij}(x)| \leq C$.

The next definition picks out subsets of $\mathcal{M}$ whose members are "uniformly bounded" away from being too large (and too small).

Definition 2.11. Let $\lambda_{\min}^{\tilde{G}}$ denote the minimal eigenvalue of $\tilde{G} = g^{-1}\tilde{g}$. A subset $\mathcal{U} \subset \mathcal{M}$ is called *amenable* if it is of the form

$$\mathcal{U} = \{\tilde{g} \in \mathcal{M} \mid \lambda_{\min}^{\tilde{G}} \geq \zeta \text{ and } |\tilde{g}_{ij}(x)| \leq C \text{ for all } \tilde{g} \in \mathcal{U},\, x \in M \text{ and } 1 \leq i, j \leq n\} \tag{2.4}$$

for some constants $C, \zeta > 0$.

We call a subset $\mathcal{U} \subset \mathcal{M}$ *quasi-amenable* if it is of the form

$$\mathcal{U} = \{\tilde{g} \in \mathcal{M} \mid |\tilde{g}_{ij}(x)| \leq C \text{ for all } \tilde{g} \in \mathcal{U},\, x \in M \text{ and } 1 \leq i, j \leq n\} \tag{2.5}$$

for some constant $C \geq 0$.

For such a subset, we denote by $\mathcal{U}^0$ its closure with respect to the L^2 norm $\|\cdot\|_g$.

Note also that (quasi-)amenable subsets are convex because of the convexity of the absolute value and the concavity of the minimal eigenvalue.

Remark 2.12. A couple of remarks about these subsets are in order:

(1) Definition 2.11 differs from the way that (quasi-)amenable subsets were defined in [3], in that here we define them to be maximal with respect to the bounds in (2.4) and (2.5). This will turn out to be the most convenient definition.
(2) We could have defined (quasi-)amenable subsets in a coordinate-independent way by replacing the condition $|\tilde{g}_{ij}| \leq C$ with the condition that for each $x \in M$, the set $|\tilde{g}(x)|_{g(x)} \leq C'$ for some other constant C'. This would have been completely equivalent for all intents and purposes, and is more satisfactory in that it does not depend on a choice of coordinate atlas. However, it would have caused the inconvenience of being incompatible with the definitions and results of [3], at least without a good deal of additional remarks at points where we use those results.

We will not need amenable subsets much in this paper, though they will come up for technical reasons at one point soon. The main point of introducing quasi-amenable subsets is that within a quasi-amenable subset, we can control the d-distance between two metrics using their distance in the fixed L^2 norm $\|\cdot\|_g$. Indeed, we have the following results, which certainly do not hold on all of $\mathcal{M}$.

Theorem 2.13 ([3, Thm. 5.12])**.** *Let $\mathcal{U} \subset \mathcal{M}$ be quasi-amenable. Then for all $\epsilon > 0$, there exists $\delta > 0$ such that if $g_0, g_1 \in \mathrm{cl}(\mathcal{U})$ (where $\mathrm{cl}(\mathcal{U})$ denotes the topological closure of $\mathcal{U} \subset \mathcal{S}$) with $\|g_0 - g_1\|_g < \delta$, then $d(g_0, g_1) < \epsilon$.*

Proposition 2.14 ([3, Prop. 5.13])**.** *Suppose $g_0 \in \mathcal{U}^0$ for some quasi-amenable subset $\mathcal{U} \subset \mathcal{M}$. Then for any sequence $\{g_k\}$ in $\mathcal{U}$ that L^2-converges to g_0, $\{g_k\}$ is d-Cauchy and there exists a subsequence $\{g_{k_l}\}$ that ω-converges to g_0.*

Note that by the discussion of the completion of $\mathcal{M}$ in Section 2.2, the Cauchy sequence $\{g_k\}$ in the above proposition is equivalent to its ω-convergent subsequence $\{g_{k_l}\}$. Thus, we have $\lim d(g_k, g_0) = \lim d(g_{k_l}, g_0) = 0$, and we get the following corollary.

Corollary 2.15. *Suppose $g_0 \in \mathcal{U}^0$ for some quasi-amenable subset $\mathcal{U} \subset \mathcal{M}$. Then for any sequence $\{g_k\}$ in $\mathcal{U}$ that L^2-converges to g_0, we have $g_k \xrightarrow{d} g_0$.*

2.4. **Properties of the metric d.** We now turn to a review of results on the behavior of d that were established in [3] and [5]. We will also need to extend some of these results to more general settings.

One extremely important aspect of the topology induced by d on $\mathcal{M}$ is the fact that the volumes of measurable subsets behave continuously, as the next theorem shows.

Theorem 2.16 ([3, Thm. 4.20])**.** *Let $\{g_k\}$ ω-converge to $g_0 \in \mathcal{M}_f$, and let $Y \subseteq M$ be any measurable subset. Then $\mathrm{Vol}(Y, g_k) \to \mathrm{Vol}(Y, g_0)$.*

Indeed, we have the following theorem, which extends [5, Lemma 12].

Lemma 2.17. *Let $g_0, g_1 \in \mathcal{M}_f$. Then for any measurable subset $Y \subseteq M$,*

$$\left| \sqrt{\mathrm{Vol}(Y, g_1)} - \sqrt{\mathrm{Vol}(Y, g_0)} \right| \leq \frac{\sqrt{n}}{4} d(g_0, g_1).$$

Proof. Let $\{g_0^k\}$ and $\{g_1^k\}$ be any sequences in $\mathcal{M}$ that ω-converge to g_0 and g_1, respectively. By [5, Lemma 12] and Theorem 2.16, we have

$$\begin{aligned}\left|\sqrt{\mathrm{Vol}(Y,g_1)}-\sqrt{\mathrm{Vol}(Y,g_0)}\right| &= \lim_{k\to\infty}\left|\sqrt{\mathrm{Vol}(Y,g_1^k)}-\sqrt{\mathrm{Vol}(Y,g_0^k)}\right| \\ &\leq \lim_{k\to\infty}\frac{\sqrt{n}}{4}d(g_0^k,g_1^k) \\ &= \frac{\sqrt{n}}{4}d(g_0,g_1).\end{aligned}$$

□

By the last theorem, the difference in the volumes of a given subset bounds the distance in d from below. Surprisingly, we also have the following result, which bounds the distance between two metrics based on the volume of the subset on which they differ.

Theorem 2.18 ([3, Thm. 4.34])**.** *Let $\mathcal{U}$ be any amenable subset with L^2-closure $\mathcal{U}^0$. Suppose that $g_0, g_1 \in \mathcal{U}^0$, and let $E := \mathrm{carr}(g_1 - g_0) = \{x \in M \mid g_0(x) \neq g_1(x)\}$. Then there exists a constant $C(n)$ depending only on $n = \dim M$ such that*

$$d(g_0,g_1) \leq C(n)\left(\sqrt{\mathrm{Vol}(E,g_0)}+\sqrt{\mathrm{Vol}(E,g_1)}\right).$$

In particular, $C(n)$ does not depend on g_0, g_1, or $\mathcal{U}$, and we have

$$\mathrm{diam}\left(\{\tilde{g} \in \mathcal{U}^0 \mid \mathrm{Vol}(M,\tilde{g}) \leq K\}\right) \leq 2C(n)\sqrt{K}.$$

Thus, the metrics g_0 and g_1 can differ *arbitrarily* on the subset E, and still their distance from one another will be uniformly bounded by the intrinsic volume of E. One consequence of this is that metrics with very small volume are close with respect to d, despite the fact that they may be geometrically very different. For example, a torus with latitudinal radius large and longitudinal radius small—a wide, thin torus, geometrically almost a circle—has small distance from a torus with both radii small—geometrically almost a point.

Another consequence of Theorem 2.18, which we mention in passing purely for its intrinsic interest, concerns certain well-known subspaces of metrics in $\mathcal{M}$.

Corollary 2.19. *With respect to d, the following submanifolds of $\mathcal{M}$ lie within a bounded region:*

1. *For μ a smooth volume form on M, the submanifold $\mathcal{M}_\mu$ of metrics inducing the volume form μ.*
2. *For $\tilde{g} \in \mathcal{M}$ any metric, the orbit of $\tilde{g}$ under the action (by pull-back) of the diffeomorphism group of M.*
3. *For $\lambda > 0$ any number, the submanifold $\mathcal{M}_\lambda$ of metrics having total volume λ.*
4. *For $\lambda > 0$, the submanifold $\mathcal{M}_\lambda^0$ of metrics having total volume less than λ.*
5. *If the base manifold M is a surface of genus $p \geq 2$, the submanifold $\mathcal{M}_{-1}$ of hyperbolic metrics on M (having constant Gaussian curvature -1).*

Furthermore, since $\mathcal{M} \cong \mathcal{M}_\lambda \times \mathbb{R}_{>0}$, we have that $\mathcal{M}$ is diffeomorphic to the product of a d-bounded subset with $\mathbb{R}_{>0}$.

Proof. The manifolds (1)–(3) clearly consist of metrics all having the same volume, and (4) consists of metrics with volume bounded above by λ. Furthermore, by the Gauß–Bonnet formula, a hyperbolic metric on a surface has total volume equal to $4\pi(p-1)$. Therefore

5 also consists of metrics all having the same volume. So the result is implied by Theorem 2.18. □

With this digression into curiosities out of the way, we return to establishing the results we need later. The next proposition extends Theorem 2.18 to the entire completion of $\mathcal{M}$.

In the proof of this proposition, and for the remainder of the paper, we denote the characteristic function of any set $E \subseteq M$ by $\chi(E)$.

Proposition 2.20. *Let $g_0, g_1 \in \mathcal{M}_f$ and $A := \operatorname{carr}(g_1 - g_0)$. Then*

$$d(g_0, g_1) \leq C(n) \left(\sqrt{\operatorname{Vol}(A, g_0)} + \sqrt{\operatorname{Vol}(A, g_1)} \right),$$

where $C(n)$ is the same constant as in Theorem 2.18.

Proof. First, for $\alpha = 0, 1$, define

$$E_\alpha^k := \left\{ x \;\middle|\; \lambda_{\min}^{G_\alpha} \geq \frac{1}{k} \text{ and } |(g_\alpha)_{ij}(x)| \leq k;\ \alpha = 0, 1 \right\},$$
$$g_\alpha^k := \chi(E_\alpha^k) g_\alpha + \chi(M \setminus E_\alpha^k) g.$$

Then there exists an amenable subset $\mathcal{U}_k$ such that $\mathcal{U}_k^0$ contains both g_0^k and g_1^k. Furthermore,

$$\operatorname{carr}(g_1^k - g_0^k) \subseteq A \cup \left(\left(E_0^k \cup E_1^k \right) \setminus \left(E_0^k \cap E_1^k \right) \right)$$

since we have only modified g_α on E_α^k, and on $E_0^k \cap E_1^k$, $g_0^k = g = g_1^k$. But we also have that if $x \notin A$, then $g_0(x) = g_1(x)$, so in this case $x \in E_0^k$ if and only if $x \in E_1^k$. In other words, $x \notin A$ implies that either $x \in E_0^k \cap E_1^k$ or $x \notin E_0^k \cup E_1^k$. From this we see that $(E_0^k \cup E_1^k) \setminus (E_0^k \cap E_1^k) \subseteq A$, implying that $\operatorname{carr}(g_1^k - g_0^k) \subseteq A$.

Thus, by Theorem 2.18, we have

$$d(g_0^k, g_1^k) \leq C(n) \left(\sqrt{\operatorname{Vol}(A, g_0^k)} + \sqrt{\operatorname{Vol}(A, g_1^k)} \right). \tag{2.6}$$

If we can now show that $g_\alpha^k \xrightarrow{\omega} g_\alpha$ for $\alpha = 0, 1$, then $\operatorname{Vol}(A, g_\alpha^k) \to \operatorname{Vol}(A, g_\alpha)$ by Theorem 2.16. This, together with (2.6), would give the result by taking the limit of both sides of the inequality.

Since $\chi(E_\alpha^k)$ converges a.e. to $\chi(M \setminus X_{g_\alpha})$ as $k \to \infty$ (recall that X_{g_α} is the deflated set of g_α), all the conditions for g_α^k to ω-converge to g_α are clear, except that we must verify that $\{g_\alpha^k\}$ is a d-Cauchy sequence.

Now, if $k, l \in \mathbb{N}$, we easily see that $E_\alpha^k \subseteq E_\alpha^{k+l}$, and that g_α^k and g_α^{k+l} only differ on $E_\alpha^{k+l} \setminus E_\alpha^k$. Furthermore, the amenable subset $\mathcal{U}_{k+l}$ can clearly be chosen such that $g_\alpha^k \in \mathcal{U}_{k+l}$. Thus, using Theorem 2.18 again, we see

$$d(g_\alpha^k, g_\alpha^{k+l}) \leq C(n) \left(\sqrt{\operatorname{Vol}(E_\alpha^{k+l} \setminus E_\alpha^k, g_\alpha^k)} + \sqrt{\operatorname{Vol}(E_\alpha^{k+l} \setminus E_\alpha^k, g_\alpha^{k+l})} \right).$$

Since on $E_\alpha^{k+l} \setminus E_\alpha^k$, we have $g_\alpha^k = g$ and $g_\alpha^{k+l} = g_\alpha$, we can rewrite the above inequality as

$$d(g_\alpha^k, g_\alpha^{k+l}) \leq C(n) \left(\sqrt{\operatorname{Vol}(E_\alpha^{k+l} \setminus E_\alpha^k, g)} + \sqrt{\operatorname{Vol}(E_\alpha^{k+l} \setminus E_\alpha^k, g_\alpha)} \right). \tag{2.7}$$

Next, note that by using the fact that $E_\alpha^{k+l} \subseteq M \setminus X_{g_\alpha}$, we can estimate

$$\operatorname{Vol}(E_\alpha^{k+l} \setminus E_\alpha^k, g_\alpha) \leq \operatorname{Vol}((M \setminus X_{g_\alpha}) \setminus E_\alpha^k, g_\alpha) \tag{2.8}$$

and

$$\mathrm{Vol}(E_\alpha^{k+l} \setminus E_\alpha^k, g) \leq \mathrm{Vol}((M \setminus X_{g_\alpha}) \setminus E_\alpha^k, g). \tag{2.9}$$

On the other hand, as we already noted, $\chi(E_\alpha^k)$ converges a.e. to $\chi(M \setminus X_{g_\alpha})$ as $k \to \infty$, and so $\chi((M \setminus X_{g_\alpha}) \setminus E_\alpha^k) \to 0$ a.e. Since we also have that $\chi((M \setminus X_{g_\alpha}) \setminus E_\alpha^k) \leq 1$ for each $k \in \mathbb{N}$, and the constant function 1 is μ_{g_α}-integrable since g_α has finite volume, we can apply the Lebesgue Dominated Convergence Theorem to see that

$$\lim_{k\to\infty} \mathrm{Vol}((M \setminus X_{g_\alpha}) \setminus E_\alpha^k, g_\alpha) = \lim_{k\to\infty} \int_M \chi(M \setminus E_\alpha^k)\, d\mu_{g_\alpha} = 0. \tag{2.10}$$

Analogously, $\lim_{k\to\infty} \mathrm{Vol}((M \setminus X_{g_\alpha}) \setminus E_\alpha^k, g) = 0$.

Combining this with (2.7), (2.8), (2.9) and (2.10) shows that for k large enough, $d(g_k^\alpha, g_{k+l}^\alpha)$ becomes arbitrarily small—independently of l—from which it follows that $\{g_k^\alpha\}$ is Cauchy, as was to be shown. □

We now have most of the prerequisite facts necessary for our study of the topology induced by d. However, we will still need to review other convergence notions, including the metric Θ_M mentioned in the introduction.

2.5. **Convergence in measure.** In the theory of L^p spaces, convergence of functions in measure plays an important role. We will need a straightforward generalization of this here, where we just replace the usual absolute value on $\mathbb{R}$ with the norm $|\cdot|_{g(x)}$.

Definition 2.21. $\{g_k\} \subset \mathcal{M}_m$ converges in (μ-)measure to $g_0 \in \mathcal{M}_m$, where μ is some Lebesgue measure on M, if for all $\epsilon > 0$,

$$\lim_{k\to\infty} \mu\left(\left\{ x \in M \;\middle|\; |g_0(x) - g_k(x)|_{g(x)} \geq \epsilon \right\}\right) = 0.$$

If μ is omitted, it is assumed that $\mu = \mu_g$, the volume form of our fixed reference metric g.

The following lemma will allow us to translate convergence in μ_g-measure to convergence in other measures.

Lemma 2.22. *Let μ be a finite Lebesgue measure on M, and let ν be another finite Lebesgue measure on M that is absolutely continuous with respect to μ. If the sequence $\{g_k\} \subset \mathcal{M}_m$ converges to $g_0 \in \mathcal{M}_m$ in μ-measure, then the sequence converges to g_0 in ν-measure as well.*

Proof. The lemma follows easily if one observes that for each $\epsilon > 0$, there exists δ such that for all measurable $E \subseteq M$, $\mu(E) < \delta$ implies $\nu(E) < \epsilon$. This, in turn, follows from the fact that

$$\lim_{C\to\infty} \mu\left(\left\{ x \in M \;\middle|\; \frac{d\nu}{d\mu}(x) \geq C \right\}\right) = 0,$$

where $d\nu/d\mu$ denotes the Radon-Nikodym derivative. □

In particular, the previous lemma applies to the case when $\mu = \mu_g$ and $\nu = \mu_{\tilde{g}}$, where $\tilde{g} \in \mathcal{M}_f$.

Finally, we need a quick definition that gives a strong type of convergence of measures that will come up later.

Definition 2.23. Let μ_k and μ be nonnegative Lebesgue measures on M. We say that $\{\mu_k\}$ *converges uniformly to* μ iff for all $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all $E \subseteq M$ measurable, $|\mu(E) - \mu_k(E)| < \epsilon$.

Remark 2.24. Notice that this convergence is stronger than, for example, weak-$*$ convergence (sometimes also just called weak convergence) of measures. It gives a topology on the space of measures on M that is equivalent to the topology induced by the supremum norm on the vector space of signed measures on M, where we set $|\mu| := \sup |\mu(E)|$, with the supremum ranging over all measurable subsets $E \subseteq M$.

The usefulness of this definition is given by its connection to the Radon-Nikodym derivative.

Lemma 2.25. *Let μ_k and μ be nonnegative Lebesgue measures on M, and let ν be any Lebesgue measure with respect to which μ and all μ_k are absolutely continuous. Furthermore, assume that $\mu_k(M), \mu(M), \nu(M) < \infty$. Then uniform convergence of $\{\mu_k\}$ to μ is equivalent to L^1-convergence of the Radon-Nikodym derivatives:*

$$\frac{d\mu_k}{d\nu} \xrightarrow{L^1(M,\nu)} \frac{d\mu}{d\nu}.$$

In order to prove this lemma, we need a characterization of convergence of L^p functions. Let (Σ, ν) be a measure space, and recall that a collection of measurable functions $\mathcal{G}$ on Σ is called *uniformly absolutely continuous* if the following holds: For all $\epsilon > 0$, there exists $\delta > 0$ such that if $E \subseteq \Sigma$ is measurable with $\nu(E) < \delta$, then

$$\int_E |f| \, d\nu < \epsilon \quad \text{for all } f \in \mathcal{G}.$$

It is not hard to see that if $\nu(\Sigma) < \infty$, then any finite set of functions is uniformly absolutely continuous.

With this definition, we have the following result.

Theorem 2.26 ([12, Thm. 8.5.14])**.** *Let (X, ν) be a measure space with $\nu(X) < \infty$, and let f be a measurable function on X. Furthermore, let f_k be a sequence of functions in $L^p(X, \nu)$. Then the following statements are equivalent.*

(1) *$f_k \to f$ in $L^p(X, \nu)$.*
(2) *$\{|f_k|^p \mid k \in \mathbb{N}\}$ is uniformly absolutely continuous and $f_k \to f$ in measure.*

We can now use this to prove the lemma.

Proof of Lemma 2.25. That L^1-convergence implies uniform convergence is a straightforward argument, so we turn to the proof of the converse statement.

By Theorem 2.26, it suffices to show that the set of functions $\{d\mu_k/d\nu \mid k \in \mathbb{N}\}$ is uniformly absolutely continuous (with respect to ν), and that $d\mu_k/d\nu$ converges to $d\mu/d\nu$ in ν-measure. (That $d\mu/d\nu$ and each $d\mu_k/d\nu$ are L^1 functions is implied by $\mu_k(M), \mu(M) < \infty$.)

To show that $\{d\mu_k/d\nu \mid k \in \mathbb{N}\}$ is uniformly absolutely continuous, let $\epsilon > 0$ be given. Since without loss of generality, we can forget a finite number of functions from the set, we may restrict to k large enough that $|\mu(E) - \mu_k(E)| < \epsilon/2$ for all measurable $E \subseteq M$. Since μ is absolutely continuous with respect to ν, there exists $\delta > 0$ such that if E is measurable and $\nu(E) < \delta$, then $\mu(E) < \epsilon/2$. So, let $Y \subseteq M$ be any measurable subset with $\nu(Y) < \delta$. Then

$$\int_Y \left| \frac{d\mu_k}{d\nu} \right| \, d\nu = \mu_k(Y) < \mu(Y) + \epsilon/2 < \epsilon,$$

showing that $\{d\mu_k/d\nu \mid k \in \mathbb{N}\}$ is uniformly absolutely continuous with respect to ν.

To see that $d\mu_k/d\nu$ converges to $d\mu/d\nu$ in ν-measure, assume the contrary. Thus, there exists $\epsilon > 0$ such that if

$$E_k^{\epsilon+} := \left\{ x \in M \;\middle|\; \frac{d\mu}{d\nu} - \frac{d\mu_k}{d\nu} \geq \epsilon \right\} \qquad \text{and} \qquad E_k^{\epsilon-} := \left\{ x \in M \;\middle|\; \frac{d\mu_k}{d\nu} - \frac{d\mu}{d\nu} \geq \epsilon \right\},$$

then

$$\limsup_{k\to\infty} \nu(E_k^{\epsilon+} \cup E_k^{\epsilon-}) = \delta > 0.$$

By additivity of ν, either $\limsup \nu(E_k^{\epsilon+}) \geq \delta/2$, or $\limsup \nu(E_k^{\epsilon-}) \geq \delta/2$. Without loss of generality, say that the former holds. This then gives that for all $k_0 \in \mathbb{N}$, there exists $k \geq k_0$ such that

$$\mu(E_k^{\epsilon+}) - \mu_k(E_k^{\epsilon+}) = \int_{E_k^{\epsilon+}} \left[\frac{d\mu}{d\nu} - \frac{d\mu_k}{d\nu} \right] d\nu \geq \epsilon \cdot \frac{\delta}{2} > 0.$$

This, however, is in direct contradiction of the assumption that μ_k converges uniformly to μ. $\square$

With these preliminaries on convergence in measure spaces out of the way, we now turn to our detailed discussion of the metric structure induced on $\mathcal{M}$ by Θ_M.

3. The metric Θ_M

3.1. **Motivation and definition.** As mentioned in the Introduction, computing d for arbitrary points $g_0, g_1 \in \mathcal{M}$ involves an infinite-dimensional problem, since we have to find the infimum of the expression

$$\begin{aligned} L(g_t) &= \int_0^1 \|g_t'\|_{g_t}\, dt = \int_0^1 \left(\int_M \operatorname{tr}_{g_t}((g_t')^2)\, d\mu_{g_t} \right)^{1/2} dt \\ &= \int_0^1 \left(\int_M \operatorname{tr}_{g_t}((g_t')^2) \sqrt{\det G_t}\, d\mu_g \right)^{1/2} dt \end{aligned} \tag{3.1}$$

over all paths g_t connecting g_0 and g_1. Furthermore, as noted at the end of Section 2.1, we cannot reduce the question to one of geodesics even for close-together points.

One solution, as hinted at in the Introduction, is changing the order of integration in (3.1). We have already taken the first step in this by removing the t-dependence from the volume form above. The second step requires introducing a new Riemannian metric on $\mathcal{M}_x$.

Definition 3.1. For each $x \in M$, define a Riemannian metric $\langle \cdot, \cdot \rangle^0$ on $\mathcal{M}_x$ by

$$\langle b, c \rangle_a^0 = \operatorname{tr}_a(bc) \det A \quad \text{for all } b, c \in T_a\mathcal{M}_x \cong \mathcal{S}_x.$$

(Recall that A denotes $g(x)^{-1}a$, cf. Convention 2.2.) We denote by θ_x^g the Riemannian distance function of $\langle \cdot, \cdot \rangle^0$.

For any measurable $Y \subseteq M$, define a function $\Theta_Y : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ by

$$\Theta_Y(g_0, g_1) = \int_Y \theta_x^g(g_0(x), g_1(x))\, d\mu_g.$$

Thus, determining $\Theta_Y(g_0, g_1)$ indeed involves finding the distance between $g_0(x)$ and $g_1(x)$ in $\mathcal{M}_x$ and then integrating this over M, as desired. Note that as θ_x^g is a Riemannian distance function on a finite-dimensional manifold, it is as usual a metric (in particular, it is positive definite).

By the results of [5, §4], we have that Θ_Y does not depend on the choice of reference metric g. Furthermore, Θ_Y is a pseudometric on $\mathcal{M}$, and in the special case $Y = M$—the one we will be most concerned with—Θ_M is a metric (in the sense of metric spaces). Using a Hölder's Inequality argument to get rid of the square root in (3.1), we also obtained the following relation between Θ_M and d [5, Prop. 27]:

$$\Theta_M(g_0, g_1) \leq d(g_0, g_1) \left(\sqrt{n}\, d(g_0, g_1) + 2\sqrt{\operatorname{Vol}(M, g_0)} \right) \tag{3.2}$$

for all $g_0, g_1 \in \mathcal{M}$. This inequality shows that the topology of Θ_M is no stronger that that of d, in the sense that $d(g_k, g_0) \to 0$ implies $\Theta_M(g_k, g_0) \to 0$ for all $g_0 \in \mathcal{M}$ and all sequences $\{g_k\} \subset \mathcal{M}$. If we had such an inequality with the roles of Θ_M and d reversed, then we would have achieved our goal of changing the order of integration in (3.1), at least as far as topological questions are concerned. Unfortunately, we do not have such an estimate, but the rest of the paper is essentially about giving us the topological results we want in a different way.

3.2. **Fundamental results on Θ_M.** In the rest of this section, we give an investigation of the properties of Θ_M, starting with a review of results we have already established elsewhere. The first such result gives an explicit description of the completion of $(\mathcal{M}_x, \theta^g_x)$, and is the first step towards understanding the completion of $(\mathcal{M}, \Theta_M)$.

Theorem 3.2 ([3, Thm. 4.14]). *For any given $x \in M$, let $\operatorname{cl}(\mathcal{M}_x)$ denote the closure of $\mathcal{M}_x \subset \mathcal{S}_x$ with regard to the natural topology. Then $\operatorname{cl}(\mathcal{M}_x)$ consists of all positive semidefinite $(0,2)$-tensors at x. Let us denote the boundary of $\mathcal{M}_x$, as a subspace of $\mathcal{S}_x$, by $\partial\mathcal{M}_x$. We denote the quotient of this closure by its boundary by $\overline{\mathcal{M}_x} := \operatorname{cl}(\mathcal{M}_x)/\partial\mathcal{M}_x$.*

Then the completion of $(\mathcal{M}_x, \theta^g_x)$ can be identified with $\overline{\mathcal{M}_x}$. The distance function is given by

$$\theta^g_x([g_0], [g_1]) = \lim_{k\to\infty} \theta^g_x(g^0_k, g^1_k), \tag{3.3}$$

where $\{g^0_k\}$ and $\{g^1_k\}$ are any sequences in $\mathcal{M}_x$ converging (in the topology of $\mathcal{S}_x$) to g_0 and g_1, respectively, in $\operatorname{cl}(\mathcal{M}_x)$.

From now on, we will drop the equivalence class notation when writing the θ^g_x-distance between elements of $\overline{\mathcal{M}_x}$, with the understanding that the formula (3.3) is implied.

Using the above theorem, we can give meaning to Θ_Y on $\overline{(\mathcal{M}, d)}$, and even extend the estimate (3.2) to this space.

Proposition 3.3 ([3, Prop. 4.25]). *Let $Y \subseteq M$ be measurable. Then the pseudometric Θ_Y on $\mathcal{M}$ can be extended to a pseudometric on $\overline{(\mathcal{M}, d)} \cong \widehat{\mathcal{M}_f}$ via*

$$\Theta_Y(\{g^0_k\}, \{g^1_k\}) := \lim_{k\to\infty} \Theta_Y(g^0_k, g^1_k) \tag{3.4}$$

for any Cauchy sequences $\{g^0_k\}$ and $\{g^1_k\}$. This pseudometric is no stronger than d in the sense that $d(\{g^0_k\}, \{g^1_k\}) = 0$ implies $\Theta_Y(\{g^0_k\}, \{g^1_k\}) = 0$. More precisely, we have

$$\Theta_Y(\{g^0_k\}, \{g^1_k\}) \leq d(\{g^0_k\}, \{g^1_k\}) \left(\sqrt{n}\, d(\{g^0_k\}, \{g^1_k\}) + 2\sqrt{\operatorname{Vol}(M, g_0)} \right),$$

where g_0 is any element of $\mathcal{M}_f$ that $\{g^0_k\}$ ω-subconverges to.

Furthermore, if $\{g_k^0\}$ *and* $\{g_k^1\}$ *are sequences in* $\mathcal{M}$ *that* ω*-converge to* g_0 *and* g_1*, respectively, then the formula*

$$\Theta_Y(\{g_k^0\}, \{g_k^1\}) = \int_Y \theta_x^g(g_0(x), g_1(x))\, \mu_g(x) \tag{3.5}$$

holds for all $g_0, g_1 \in \mathcal{M}$*.*

In view of the formula (3.5), we will from now on write simply $\Theta_Y(g_0, g_1)$ for any $g_0, g_1 \in \mathcal{M}_f$, where it is understood that this quantity is given by (3.4) or, equivalently, (3.5).

The next result we will make use of gives the pointwise version of Lemma 2.17.

Lemma 3.4 ([3, Lemma 4.10])**.** *Let* $a_0, a_1 \in \mathcal{M}_x$*. Then*

$$\left| \sqrt{\det A_1} - \sqrt{\det A_0} \right| \leq \frac{\sqrt{n}}{2} \theta_x^g(a_0, a_1).$$

It will be necessary for us to make a straightforward extension of this result to $\mathrm{cl}(\mathcal{M}_x)$ using (3.3).

Lemma 3.5. *Let* $a_0, a_1 \in \mathrm{cl}(\mathcal{M}_x)$*. Then*

$$\left| \sqrt{\det A_1} - \sqrt{\det A_0} \right| \leq \frac{\sqrt{n}}{2} \theta_x^g(a_0, a_1). \tag{3.6}$$

Proof. Because of Lemma 3.4, it only remains to deal with the case that at least one of a_0 or a_1 belongs to $\partial\mathcal{M}_x$.

If both belong to $\partial\mathcal{M}_x$, then both sides of (3.6) are zero, so there is nothing to prove.

We are left with the case that only one belongs to $\partial\mathcal{M}_x$ (let's say, without loss of generality, that it's a_0). Let $\{a_0^k\}$ be a sequence in $\mathcal{M}_x$ that θ_x^g-converges to a_0. Then $\det A_0^k \to \det A_0 = 0$ by Theorem 3.2, and we also have

$$\left| \sqrt{\det A_1} - \sqrt{\det A_0^k} \right| \leq \frac{\sqrt{n}}{2} \theta_x^g(a_0^k, a_1).$$

Taking the limit as $k \to \infty$ of both sides gives the result. □

We can then integrate the estimate of the last lemma to get an analogous result for Θ_M.

Lemma 3.6. *Let* $Y \subseteq M$ *be measurable. Then the function* $\mathrm{Vol}(Y, \cdot\,) : \mathcal{M}_f \to \mathbb{R}$ *mapping* $\tilde{g} \mapsto \mathrm{Vol}(M, \tilde{g})$ *is Lipschitz continuous with respect to* Θ_Y*. In particular, if* $g_0, g_1 \in \mathcal{M}_f$*, then*

$$|\mathrm{Vol}(M, g_1) - \mathrm{Vol}(M, g_0)| \leq \frac{\sqrt{n}}{2} \Theta_Y(g_0, g_1) \leq \frac{\sqrt{n}}{2} \Theta_M(g_0, g_1).$$

Proof. By [3, Lemma 4.10], we have that

$$\left| \sqrt{\det G_1(x)} - \sqrt{\det G_0(x)} \right| \leq \frac{\sqrt{n}}{2} \theta_x^g(g_0(x), g_1(x)).$$

Using this, we can estimate

$$\begin{aligned}
|\mathrm{Vol}(M, g_1) - \mathrm{Vol}(M, g_0)| &= \left| \int_Y d\mu_{g_1} - \int_Y d\mu_{g_0} \right| \\
&= \left| \int_Y \left(\sqrt{\det G_1(x)} - \sqrt{\det G_0(x)} \right) d\mu_g \right| \\
&\leq \int_Y \left| \sqrt{\det G_1(x)} - \sqrt{\det G_0(x)} \right| d\mu_g \\
&\leq \frac{\sqrt{n}}{2} \int_Y \theta_x^g(g_0(x), g_1(x))\, d\mu_g \\
&= \frac{\sqrt{n}}{2} \Theta_Y(g_0, g_1) \leq \frac{\sqrt{n}}{2} \Theta_M(g_0, g_1).
\end{aligned}$$

□

3.3. **The completion of** $(\mathcal{M}, \Theta_M)$**.** The above lemma suggests a strong parallel with the behavior of d—again, compare Lemma 2.17. In fact, the next two theorems will give us even stronger parallels, as we will see that the completion of $(\mathcal{M}, \Theta_M)$ can be identified with a quotient space of $\mathcal{M}_f$. We begin with a proof that to each Θ_M-Cauchy sequence, we can associate a limit semimetric in $\mathcal{M}_m$.

Before we prove these theorems, let us first remark that for a sequence $\{g_k\} \subset \mathcal{M}_m$, we define θ_x^g-convergence in measure to $g_0 \in \mathcal{M}_m$ analogously to how we defined it for $|\cdot|_{g(x)}$. Again, if the measure is not explicitly mentioned, then μ_g is implied.

Theorem 3.7. *Let $\{g_k\} \subset \mathcal{M}_f$ be a Θ_M-Cauchy sequence. Then there exists an element $[g_0] \in \widehat{\mathcal{M}_m}$ such that $g_k \xrightarrow{\Theta_M} [g_0]$. (In particular, $\Theta_M(g_k, [g_0])$ is well-defined and finite for each $k \in \mathbb{N}$.) Furthermore, if $g_0 \in [g_0]$ is any representative, then we have that $\{g_k\}$ θ_x^g-converges to g_0 in measure, and $X_{g_0} = D_{\{g_k\}}$ (cf. Definition 2.4) up to a nullset.*

Finally, if $\{\tilde{g}_k\} \subset \mathcal{M}_f$ is any Θ_M-Cauchy sequence that θ_x^g-converges to $\tilde{g}_0 \in \mathcal{M}_m$ in measure, then $\tilde{g}_k \xrightarrow{\Theta_M} \tilde{g}_0$.

Proof. We begin with the first statement. Since $\{g_k\}$ is a Θ_M-Cauchy sequence, it suffices to prove convergence for a subsequence. By passing to a subsequence, we may assume that

$$\sum_{k=1}^{\infty} \Theta_M(g_k, g_{k+1}) < \infty. \tag{3.7}$$

Now, for all $\delta > 0$ and $k, l \in \mathbb{N}$, let $E_\delta^{k,l} := \{x \in M \mid \theta_x^g(g_k, g_l) \geq \delta\}$. For all $\epsilon > 0$, we can find $k_0 \in \mathbb{N}$ such that if $k, l \geq k_0$, then $\mathrm{Vol}(E_\delta^{k,l}, g) < \epsilon$. (In other words, $\{g_k\}$ is θ_x^g-Cauchy in measure.) A straightforward argument shows that by again passing to a subsequence, we can assume that $\{g_k\}$ is θ_x^g-Cauchy a.e. Therefore, by Theorem 3.2, for almost every $x \in M$, $\{g_k(x)\}$ θ_x^g-converges to some element $[a_x] \in \overline{\mathcal{M}_x}$. For each $x \in M$, let $g_0(x) := a_x$, where $a_x \in [a_x]$ is any representative. (Note that g_0 is well-defined up to a set of measure zero, where we may without consequence set it equal to zero.) We claim that $[g_0]$ is the desired limit element.

Choose any representative $g_0 \in [g_0]$. (Note that the choice of representative does not affect the quantity $\Theta_M(g_k, g_0)$.) Then for a.e. $x \in M$ and $k \in \mathbb{N}$, $\theta_x^g(g_k(x), g_0(x))$ is finite, positive, and independent of our choice of the representative g_0. Fix some $k \in \mathbb{N}$. For each $l \in \mathbb{N}$,

define functions by $f_{k,l}(x) := \theta^g_x(g_k(x), g_l(x))$, and also define $f_k(x) := \theta^g_x(g_k(x), g_0(x))$. Then by construction, we have that $\lim_{l\to\infty} f_{k,l}(x) = f_k(x)$ for a.e. $x \in M$. Furthermore, if we define

$$\alpha(x) := \sum_{m=1}^{\infty} \theta^g_x(g_m, g_{m+1}),$$

then by the triangle inequality,

$$|f_{k,l}(x)| \leq \sum_{m=k}^{l-1} \theta^g_x(g_m, g_{m+1}) \leq \alpha(x).$$

On the other hand, we claim that $\alpha \in L^1(M, g)$, since

$$\int_M \alpha \, d\mu_g = \int_M \sum_{m=1}^{\infty} \theta^g_x(g_m, g_{m+1}) \, d\mu_g = \sum_{m=1}^{\infty} \Theta_M(g_m, g_{m+1}) < \infty.$$

(Note here that we have used the assumption (3.7), and that we have implicitly exchanged an infinite sum and an integral in the second equality. The latter is justified by an application of the Monotone Convergence Theorem of Lebesgue and Levi [1, Thm. 2.8.2]—see the proof of Lemma 4.17 in [4] for the full details of this argument.)

Since for a.e. $x \in M$, $\lim_{l\to\infty} f_{k,l}(x) = f_k(x)$, $|f_{k,l}(x)| \leq \alpha(x)$, and $\alpha \in L^1(M, g)$, the Lebesgue Dominated Convergence Theorem applies to give that $f_k \in L^1(M, g)$ and

$$\int_M f_k \, d\mu_g = \lim_{l\to\infty} \int_M f_{k,l} \, d\mu_g.$$

In other words, $\lim_{l\to\infty} \Theta_M(g_k, g_l) = \Theta_M(g_k, g_0) < \infty$. In particular, $\Theta_M(g_k, g_0)$ is well-defined and finite, as claimed in the theorem.

To see that $\lim_{k\to\infty} \Theta_M(g_k, g_0) = 0$, one must apply essentially the same argument to the sequence of functions $f_k(x) = \theta^g_x(g_k(x), g_0(x))$ from above with the limit function 0. The function that bounds each $f_k(x)$ is

$$\theta^g_x(g_0(x), g_1(x)) + \sum_{m=1}^{\infty} \theta^g_x(g_m(x), g_{m+1}(x)),$$

which we claim is in $L^1(M, g)$. For, by a special case of the preceding argument, $\Theta_M(g_1, g_0) < \infty$, which is equivalent to saying that the first term is L^1. But this already implies the claim, since the sum is just $\alpha(x)$, which we saw was L^1 in the preceding argument.

Thus, we now have that $g_k \xrightarrow{\Theta_M} [g_0]$, or in other words, $\theta^g_x(g_k(x), g_0(x)) \xrightarrow{L^1(M,g)} 0$, implying that $\theta^g_x(g_k(x), g_0(x)) \to 0$ in measure by Theorem 2.26. But this is exactly the assertion that $\{g_k\}$ θ^g_x-converges to g_0 in measure. From this, and [3, Lemma 4.10], one can also deduce that $D_{\{g_k\}} = X_{g_0}$ up to a nullset.

Finally, let $\{\tilde{g}_k\} \subset \mathcal{M}_f$ be any Θ_M-Cauchy sequence that θ^g_x-converges to $\tilde{g}_0 \in \mathcal{M}_m$ in measure. We know that $\{\tilde{g}_k\}$ Θ_M-converges to some limit $\bar{g}_0 \in \mathcal{M}_m$, and that $\{\tilde{g}_k\}$ θ^g_x-converges to $\bar{g}_0$ in measure. So it is not hard to see that off of the set where both $\tilde{g}_0(x)$ and $\bar{g}_0(x)$ are degenerate (that is, off of the set where it is possible that $\tilde{g}_0(x) \neq \bar{g}_0(x)$ but $\theta^g_x(\tilde{g}_0(x), \bar{g}_0(x)) = 0$), $\tilde{g}_0$ and $\bar{g}_0$ must coincide a.e. In other words, $[\tilde{g}_0] = [\bar{g}_0]$. This implies that $\Theta_M(\tilde{g}_k, \tilde{g}_0) = \Theta_M(\tilde{g}_k, \bar{g}_0)$, and so $\tilde{g}_k \to \tilde{g}_0$, as was to be shown. $\square$

Knowing now that the Θ_M-limit of a Cauchy sequence can be identified with an element of $\mathcal{M}_m$, we demonstrate that this limit must have finite volume, and thus actually lies in $\mathcal{M}_f$.

Theorem 3.8. *Let $\{g_k\} \subset \mathcal{M}_f$ Θ_M-converge to $g_0 \in \mathcal{M}_m$. Then in fact $g_0 \in \mathcal{M}_f$ and the following hold:*

(1) *We have*
$$\left(\frac{\mu_{g_k}}{\mu_g}\right) \xrightarrow{L^1(M,g)} \left(\frac{\mu_{g_0}}{\mu_g}\right).$$
In particular, $\{(\mu_{g_\alpha}/\mu_g) \mid \alpha = 0, 1, 2, \dots\}$ is uniformly absolutely continuous with respect to μ_g.

(2) *μ_{g_k} converges uniformly to μ_{g_0}.*

Proof. We first prove statement (1). By Lemma 3.5, we have
$$\int_M \left| \left(\frac{\mu_{g_0}}{\mu_g}\right) - \left(\frac{\mu_{g_k}}{\mu_g}\right) \right| d\mu_g \leq \frac{\sqrt{n}}{2} \int_M \theta^g_x(g_k(x), g_0(x))\, d\mu_g = \Theta_M(g_k, g_0).$$
The statement follows from this immediately. Note that this also implies that we haveo $(\mu_{g_0}/\mu_g) \in L^1(M, g)$, or in other words, g_0 has finite total volume. The uniform absolute continuity of $\{(\mu_{g_\alpha}/\mu_g) \mid \alpha = 0, 1, 2, \dots\}$ is given by Theorem 2.26.

Statement (2) then follows from statement (1) and Lemma 2.25. $\square$

The above result implies, as noted, that $\overline{(\mathcal{M}, \Theta_M)}$ is a quotient space of some subspace of $\mathcal{M}_f$. In fact, it is the same space as $\overline{(\mathcal{M}, d)}$:

Theorem 3.9. *The completion of $\mathcal{M}$ with respect to Θ_M can be naturally identified with $\widehat{\mathcal{M}_f}$. This map is an isometry if we define Θ_M on $\widehat{\mathcal{M}_f}$ by*
$$\Theta_M([g_0], [g_1]) = \int_M \theta^g_x(g_0(x), g_1(x))\, d\mu_g$$
for any $g_0, g_1 \in \mathcal{M}_f$.

Proof. By Theorems 3.7 and 3.8, any Θ_M-Cauchy sequence in $\mathcal{M}$ Θ_M-converges to an element of $\mathcal{M}_f$. Furthermore, if $g_0 \in \mathcal{M}_f$ is any element, then there exists a d-Cauchy sequence that d-converges to g_0 (cf. Theorem 2.6). By the estimate of Proposition 3.3, one can see that this sequence is also Θ_M-Cauchy. Thus, any element of $\mathcal{M}_f$ arises as the Θ_M-limit of some sequence in $\mathcal{M}$. This proves that $\overline{(\mathcal{M}, \Theta_M)}$ is a quotient space of $\mathcal{M}_f$ given by identifying all elements with distance zero from one another. Furthermore, Proposition 3.3 implies the formula
$$\Theta_M(g_0, g_1) = \int_M \theta^g_x(g_0(x), g_1(x))\, d\mu_g \tag{3.8}$$
for any $g_0, g_1 \in \mathcal{M}_f$.

But from (3.8) and the fact that θ^g_x is a metric on $\mathcal{M}_x$ and zero on $\partial\mathcal{M}_x$, it is easy to see that if $g_0, g_1 \in \mathcal{M}_f$, then $g_0 \sim g_1$ if and only if $\Theta_M(g_0, g_1) = 0$. Thus the desired quotient space of $\mathcal{M}_f$ is exactly $\widehat{\mathcal{M}_f} = \mathcal{M}_f/{\sim}$. $\square$

We now have a good understanding of the metric Θ_M, but before we leave this section, we will prove a result that will come in useful later. It is based on the following pointwise result, which was already known.

Proposition 3.10 ([3, Prop. 4.13]). *Let $a_0, a_1 \in \mathcal{M}_x$. Then there exists a constant $C'(n)$, depending only on n, such that*

$$\theta_x^g(a_0, a_1) \leq C'(n) \left(\sqrt{\det A_0} + \sqrt{\det A_1} \right).$$

By integrating this inequality, we get an estimate for Θ_M that is of exactly the same form as Proposition 2.20.

Proposition 3.11. *Let $g_0, g_1 \in \mathcal{M}_f$, and let $A := \operatorname{carr}(g_1 - g_0)$. Then there exists a constant $C'(n)$, depending only on $n = \dim M$, such that*

$$\Theta_M(g_0, g_1) \leq C'(n) \left(\operatorname{Vol}(A, g_0) + \operatorname{Vol}(A, g_1) \right). \tag{3.9}$$

Proof. We claim that the inequality of Proposition 3.10 still holds if $a_0, a_1 \in \operatorname{cl}(\mathcal{M}_x)$. For if both lie in $\partial \mathcal{M}_x$, then by Theorem 3.2 we have that $\theta_x^g(a_0, a_1) = 0$, so the statement is vacuous. If one (say a_0) is in $\partial \mathcal{M}_x$, then we choose a sequence $a_0^k \xrightarrow{\theta_x^g} a_0$, where all $a_0^k \in \mathcal{M}_x$. By Lemma 3.5, $\sqrt{\det A_0^k} \to \sqrt{\det A_0} = 0$. On the other hand, for each $k \in \mathbb{N}$ we have

$$\theta_x^g(a_0^k, a_1) \leq C'(n) \left(\sqrt{\det A_0^k} + \sqrt{\det A_1} \right),$$

so taking the limit of the above inequality proves the claim.

Finally, we note that since $\sqrt{\det G_0} = (\mu_{g_0}/\mu_g)$ and $\sqrt{\det G_1} = (\mu_{g_1}/\mu_g)$, the inequality (3.9) follows immediately from integrating the pointwise estimate of Proposition 3.10 after substituting $a_0 := g_0(x)$ and $a_1 := g_1(x)$. □

We now have all of the background results that we need and are ready to move into the main body of the paper.

4. Convergence results

In this section, we will begin by proving the equivalence of the topologies of d and Θ_M on a given quasi-amenable subset. We will then use this to extend the equivalence to all of $\mathcal{M}_f$. Using that, we will finally relatively quickly arrive at a homeomorphism between $\overline{(\mathcal{M}, d)}$ and $\overline{(\mathcal{M}, \Theta_M)}$.

4.1. The topology on quasi-amenable subsets.

We begin this subsection with a straightforward extension of Theorem 2.13 to degenerate metrics.

Lemma 4.1. *Let a quasi-amenable subset $\mathcal{U} \subset \mathcal{M}$ and $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that if $g_0, g_1 \in \mathcal{U}^0$ (the L^2-closure of $\mathcal{U}$) with $\|g_1 - g_0\|_g < \delta$, we have $d(g_0, g_1) < \epsilon$.*

Proof. Let $\delta > 0$ be the number, guaranteed by Theorem 2.13, for which the following holds: For all $\tilde{g}_0, \tilde{g}_1 \in \mathcal{U}$ with $\|\tilde{g}_1 - \tilde{g}_0\|_g < 2\delta$, we have $d(\tilde{g}_0, \tilde{g}_1) < \epsilon$.

We claim that this is the desired number δ. To see this, let $g_0, g_1 \in \mathcal{U}^0$ with $\|g_1 - g_0\|_g < \delta$, and choose sequences $\{g_0^k\}$ and $\{g_1^k\}$ in $\mathcal{U}$ that both L^2- and d-converge to g_0 and g_1, respectively. (The existence of such sequences is assured by Proposition 2.14.) Then by definition,

$$d(g_0, g_1) = \lim_{k \to \infty} d(g_0^k, g_1^k). \tag{4.1}$$

On the other hand, since $g_i^k \xrightarrow{L^2} g_i$ for $i = 0, 1$, we have

$$\lim_{k\to\infty} \left\| g_1^k - g_0^k \right\|_g = \| g_1 - g_0 \|_g .$$

Since $\| g_1 - g_0 \|_g < \delta$, this implies that $\left\| g_1^k - g_0^k \right\|_g < 2\delta$ for k large enough, and so by the assumption on δ, $d(g_0^k, g_1^k) < \epsilon$ for k large. But then (4.1) implies that $d(g_0, g_1) < \epsilon$, as was to be proved. ☐

Next, we need a lemma that allows us to compare open balls in the metric θ_x^g to open balls in the norm $|\cdot|_{g(x)}$. It is possible to do this uniformly if we restrict to compact subsets in $\mathcal{M}_x$; in particular, we will need those subsets given in the next lemma.

Lemma 4.2. *Let any numbers $\zeta, \tau > 0$ be given. For each $x \in M$, we define*

$$\mathcal{M}_x^{\zeta,\tau} := \{ a \in \mathcal{M}_x \mid \sqrt{\det A} \geq \zeta,\ |a_{ij}| \leq \tau \text{ for all } 1 \leq i, j \leq n \} \subset \mathcal{M}_x.$$

Furthermore, for any $\lambda \geq 0$ and $a \in \mathcal{M}_x$, denote by $B_a^{\theta_x^g}(\lambda)$ and $B_a^{|\cdot|_{g(x)}}(\lambda)$ the open balls of radius λ around a with respect to θ_x^g and $|\cdot|_{g(x)}$, respectively.

For each $x \in M$, $a \in \mathcal{M}_x$, and $\kappa > 0$, we also define a function

$$\eta_{x,a}(\kappa) := \sup \left\{ \lambda \in \mathbb{R} \;\middle|\; B_a^{\theta_x^g}(\lambda) \subset B_a^{|\cdot|_{g(x)}}(\kappa) \right\}.$$

Then $\eta_{x,a}$ takes values in $(0, \infty)$, as does the function

$$\eta(\kappa) := \inf_{x \in M,\ a \in \mathcal{M}_x^{\zeta,\tau}} \eta_{x,a}(\kappa). \tag{4.2}$$

Proof. Since $\mathcal{M}_x$ is a finite-dimensional manifold, the topologies induced by $|\cdot|_{g(x)}$ and θ_x^g coincide. This implies, in particular, that for all $\kappa > 0$ and $a \in \mathcal{M}_x$, we can find $\lambda > 0$ such that

$$B_a^{\theta_x^g}(\lambda) \subset B_a^{|\cdot|_{g(x)}}(\kappa).$$

This also implies that the supremum of such λ must be finite, proving that $\eta_{x,a}$ takes values in $(0, \infty)$.

To see that η is also a positive, finite function, we note that $\langle \cdot, \cdot \rangle_a^0$ and $\langle \cdot, \cdot \rangle_{g(x)}$ depend smoothly on x and a. Thus, $\eta_{x,a}$ is continuous separately in x and a. Therefore the result follows from the compactness of $\mathcal{M}_x^{\zeta,\tau}$ and M. ☐

Using this lemma, we can get a bound on the distance between two elements of a quasi-amenable subset if we have a uniform, pointwise bound on their distance in θ_x^g. This, of course, will not help us much when trying to prove d-convergence of a sequence $\{g_k\}$ that Θ_M-converges to g_0, as such a sequence only has $\theta_x^g(g_k(x), g_0(x))$ converging to zero in $L^1(M, g)$. However, it will be sufficiently strong to facilitate a cut-off argument. These arguments will be a recurring theme in the remaining proofs.

Lemma 4.3. *Let $\mathcal{U} \subset \mathcal{M}$ be any quasi-amenable subset, and let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that if $g_0 \in \mathcal{U}^0$, $g_1 \in \mathcal{M}_f$, and $\theta_x^g(g_0(x), g_1(x)) < \delta$ for all $x \in M$, we have $d(g_0, g_1) < \epsilon$.*

Proof. Let $\epsilon > 0$ be given. The idea is to cut off g_0 and g_1 by setting them equal to zero on a small subset such that the cut-off semimetrics belong to a common quasi-amenable subset. The distance from the cut-off semimetrics to the original semimetrics will then be small by

Proposition 2.20. The distance in the L^2 norm between the two cut-off semimetrics can be estimated using Lemma 4.2, which gives us a bound on the d-distance by Lemma 4.1.

To fill in the details of this, we first define a positive constant ζ by

$$\zeta := \frac{\epsilon^2}{32C(n)^2 \operatorname{Vol}(M,g)} \tag{4.3}$$

where $C(n)$ is the constant from Proposition 2.20.

Let τ be the number such that the quasi-amenable subset $\mathcal{U}$ is given by

$$\{g \in \mathcal{M} \mid |g_{ij}(x)| \leq \tau \text{ for all } 1 \leq i,j \leq n,\ x \in M\}.$$

As in the last lemma, for each $x \in M$, we then consider the set

$$\mathcal{M}_x^{\zeta,\tau} := \{a \in \mathcal{M}_x \mid \sqrt{\det A} \geq \zeta,\ |a_{ij}| \leq \tau \text{ for all } 1 \leq i,j \leq n\} \subset \mathcal{M}_x,$$

We also define the positive function η as in the last lemma.

Next, we denote by $\tilde{\mathcal{U}}$ the "double" of $\mathcal{U}$, i.e.,

$$\tilde{\mathcal{U}} := \{g \in \mathcal{M} \mid |g_{ij}(x)| \leq 2\tau \text{ for all } 1 \leq i,j \leq n,\ x \in M\}.$$

Then it is clear that there exists $\alpha > 0$ such that if $\tilde{g}_0 \in \mathcal{U}^0$, $\tilde{g}_1 \in \mathcal{M}_f$, and

$$|\tilde{g}_1(x) - \tilde{g}_0(x)|_{g(x)} < \alpha \quad \text{for all } x \in M,$$

then $\tilde{g}_1 \in \tilde{\mathcal{U}}^0$.

Let κ now be the constant from Lemma 4.1 such that $\tilde{g}_0, \tilde{g}_1 \in \tilde{\mathcal{U}}^0$ and $\|\tilde{g}_1 - \tilde{g}_0\|_g < \kappa$ implies $d(\tilde{g}_0, \tilde{g}_1) < \epsilon/2$. Then we claim that

$$\delta = \min\left\{\eta(\alpha), \eta\left(\frac{\kappa}{\sqrt{\operatorname{Vol}(M,g)}}\right), \frac{\epsilon^2}{16\sqrt{n}C(n)^2 \operatorname{Vol}(M,g)}\right\} \tag{4.4}$$

is the desired constant.

To see this, let $g_0, g_1 \in \mathcal{U}^0$ with $\theta_x^g(g_0(x), g_1(x)) < \delta$ be given. Furthermore, let

$$E^\zeta := \left\{x \in M \mid g_0(x) \in \mathcal{M}_x^{\zeta,\tau}\right\}.$$

We then define

$$g_0^\zeta := \chi(E^\zeta) g_0 \quad \text{and} \quad g_1^\zeta := \chi(E^\zeta) g_1.$$

Since $(\mu_{g_0}/\mu_g) < \zeta$ on $M \setminus E^\zeta$, we have that

$$\operatorname{Vol}(M \setminus E^\zeta, g_0) = \int_{E^\zeta} \left(\frac{\mu_{g_0}}{\mu_g}\right) d\mu_g < \zeta \int_{E^\zeta} d\mu_g \leq \zeta \cdot \operatorname{Vol}(M,g).$$

Additionally, since $\theta_x^g(g_0(x), g_1(x)) < \delta$ for all $x \in M$, $\Theta_M(g_0, g_1) < \delta \cdot \operatorname{Vol}(M,g)$. Therefore, by Lemma 3.6,

$$\operatorname{Vol}(M \setminus E^\zeta, g_1) \leq \operatorname{Vol}(M \setminus E^\zeta, g_0) + \frac{\sqrt{n}}{2}\Theta_M(g_0,g_1) < \zeta \cdot \operatorname{Vol}(M,g) + \frac{\sqrt{n}}{2}\delta \cdot \operatorname{Vol}(M,g).$$

By Proposition 2.20, and using that g_0 and g_0^ζ differ only on $M \setminus E^\zeta$, where $g_0^\zeta = 0$, we thus see that

$$d(g_0, g_0^\zeta) \leq C(n)\sqrt{\operatorname{Vol}(M \setminus E^\zeta, g_0)} < C(n)\sqrt{\zeta \cdot \operatorname{Vol}(M,g)}.$$

Similarly,

$$d(g_1, g_1^\zeta) \leq C(n)\sqrt{\operatorname{Vol}(M \setminus E^\zeta, g_1)} < C(n)\sqrt{\zeta \cdot \operatorname{Vol}(M,g) + \frac{\sqrt{n}}{2}\delta \cdot \operatorname{Vol}(M,g)}.$$

Inserting (4.3) and (4.4) (for δ, use the last of the values in the minimum) into the above estimate and simplifying gives

$$d(g_0, g_0^\zeta) + d(g_1, g_1^\zeta) < \epsilon/2. \tag{4.5}$$

We now wish to estimate $d(g_0^\zeta, g_1^\zeta)$, so that we can use (4.5) and the triangle inequality to estimate $d(g_0, g_1)$. To do so, note that g_0^ζ and g_1^ζ differ only on E^ζ, where $g_0(x) \in \mathcal{M}_x^{\zeta,\tau}$. But we also know that

$$\theta_x^g(g_0^\zeta(x), g_1^\zeta(x)) < \delta \leq \eta(\alpha),$$

meaning

$$\left| g_1^\zeta(x) - g_0^\zeta(x) \right|_{g(x)} < \alpha.$$

By our choice of α, this implies that $g_1^\zeta \in \tilde{\mathcal{U}}^0$.

Finally, we recall that

$$\theta_x^g(g_0^\zeta(x), g_1^\zeta(x)) < \delta \leq \eta\left(\frac{\kappa}{\sqrt{\operatorname{Vol}(M, g)}} \right)$$

for all $x \in M$. By the definition of η, this immediately implies that

$$\left| g_1^\zeta(x) - g_0^\zeta(x) \right|_{g(x)} < \frac{\kappa}{\sqrt{\operatorname{Vol}(M, g)}}.$$

Therefore,

$$\left\| g_1^\zeta - g_0^\zeta \right\|_g = \left(\int_M \left| g_1^\zeta(x) - g_0^\zeta(x) \right|_{g(x)}^2 \, d\mu_g \right)^{1/2} < \left(\frac{\kappa^2}{\operatorname{Vol}(M, g)} \int_M \, d\mu_g \right)^{1/2} = \kappa.$$

But by our choice of κ, and since as we noted, $g_0^\zeta, g_1^\zeta \in \tilde{\mathcal{U}}^0$, the above inequality implies that $d(g_0^\zeta, g_1^\zeta) < \epsilon/2$. This, combined with (4.5) and the triangle inequality, gives the desired result. □

With this long estimate out of the way, we can show the equivalence of the topologies of d and Θ_M on a quasi-amenable subset. In fact, the following proposition is even more general, as it only requires that the limit semimetric be bounded—i.e., lie in some quasi-amenable subset. The sequence converging to this limit can have elements anywhere in $\mathcal{M}_f$.

Proposition 4.4. *Let $g_0 \in \mathcal{M}_f$ be a bounded semimetric, and let $\{g_k\} \subset \mathcal{M}_f$ be any sequence. Then $g_k \xrightarrow{d} g_0$ if and only if $g_k \xrightarrow{\Theta_M} g_0$.*

Proof. That $g_k \xrightarrow{d} g_0$ implies $g_k \xrightarrow{\Theta_M} g_0$ is clear from [3, Prop. 4.25], so we turn to the converse statement.

Let $\epsilon > 0$ be given, and let $\delta > 0$ be the number guaranteed by Lemma 4.3. We then define

$$E_k^\delta := \{ x \mid \theta_x^g(g_k(x), g_0(x)) \geq \delta \}$$

and

$$g_k^\delta := \chi(E_k^\delta) g_0 + \chi(M \setminus E_k^\delta) g_k.$$

The first thing we see is that g_k^δ differs from g_0 only on $M \setminus E_k^\delta$, and that for $x \in M \setminus E_k^\delta$, we have $\theta_x^g(g_k^\delta(x), g_0(x)) < \delta$. Therefore, by Lemma 4.3 and the choice of δ, we immediately get

$$d(g_k^\delta, g_0) < \epsilon. \tag{4.6}$$

(Note that the boundedness of g_0 implies that there exists a quasi-amenable subset $\mathcal{U}$ with $g_0 \in \mathcal{U}^0$, so that Lemma 4.3 indeed applies.)

We now claim that

$$\begin{aligned} d(g_k, g_k^\delta) &\leq C(n) \left(\sqrt{\operatorname{Vol}(E_k^\delta, g_k)} + \sqrt{\operatorname{Vol}(E_k^\delta, g_k^\delta)} \right) \\ &= C(n) \left(\sqrt{\operatorname{Vol}(E_k^\delta, g_k)} + \sqrt{\operatorname{Vol}(E_k^\delta, g_0)} \right). \end{aligned} \tag{4.7}$$

The first line follows from Proposition 2.20. The second line follows because g_k^δ coincides with g_0 on E_k^δ.

Now, because of Lemma 3.6, we can choose k large enough that $\operatorname{Vol}(E_k^\delta, g_k) \leq \operatorname{Vol}(E_k^\delta, g_0) + \epsilon^2$. Furthermore, by Theorem 2.26, $g_k \xrightarrow{\Theta_M} g_0$ implies that the function $\theta_x^g(g_k(x), g_0(x))$ on M converges to zero in μ_g-measure. By Lemma 2.22, the convergence also holds in μ_{g_0}-measure, so we can choose k large enough that $\operatorname{Vol}(E_k^\delta, g_0) < \epsilon^2$. This together with (4.7) implies that for k large enough,

$$d(g_k, g_k^\delta) \leq C(n) \cdot (\sqrt{2} + 1)\epsilon. \tag{4.8}$$

Thus, (4.6), (4.8), and the triangle inequality imply that for k large enough,

$$d(g_k, g_0) < C(n) \cdot (\sqrt{2} + 2)\epsilon.$$

Since ϵ was arbitrary, the fact that $d(g_k, g_0) \to 0$ follows. ☐

4.2. **The topology on $\mathcal{M}_f$ and the completions of $\mathcal{M}$.** The next goal is to use the results of the last subsection to extend Proposition 4.4 to allow the limit semimetric to be any element of $\mathcal{M}_f$. To do so, we will use a strategy similar to that of the proof of [3, Thm. 5.14]. The basic idea is to reduce the case of an unbounded limit semimetric to that of a bounded one by multiplying the sequence in question and its limit with appropriate positive functions that tame the unbounded parts of the limit semimetric. If we do this carefully, then we can use some arguments taking advantage of Proposition 2.1 to control how far (with respect to d) these conformal changes move the sequence and limit within $\mathcal{M}_f$.

To begin with, let's investigate how conformal changes affect the Θ_M-distance between points.

Lemma 4.5. *Let $g_0, g_1 \in \mathcal{M}_f$, and let ρ be a measurable, positive function with $\rho(x) \leq 1$ for all $x \in M$. Then $\Theta_M(\rho g_0, \rho g_1) \leq \Theta_M(g_0, g_1)$.*

Proof. The statement would follow immediately if we showed that $\theta_x^g(\rho g_0(x), \rho g_1(x)) \leq \theta_x^g(g_0(x), g_1(x))$ for all $x \in M$. So fix an arbitrary $x \in M$ and $\epsilon > 0$, and let a_t, for $t \in [0,1]$, be any path in $\mathcal{M}_x$ with endpoints $g_0(x)$ and $g_1(x)$ such that

$$L^0(a_t) \leq \theta_x^g(g_0(x), g_1(x)) + \epsilon,$$

where $L^0(a_t)$ denotes the length of a_t as measured by $\langle \cdot, \cdot \rangle^0$.

Since $\rho(x)a_t$ is a path from $\rho(x)g_0(x)$ to $\rho(x)g_1(x)$, we have

$$\begin{aligned}\theta_x^g(\rho(x)g_0(x),\rho(x)g_1(x)) \leq L^0(\rho(x)a_t) &= \int_0^1 \sqrt{\langle \rho(x)a_t', \rho(x)a_t'\rangle^0_{\rho(x)a_t}}\, dt \\ &= \int_0^1 \sqrt{\operatorname{tr}_{\rho(x)a_t}((\rho(x)a_t')^2)\det(\rho(x)A_t)}\, dt \\ &= \int_0^1 \rho(x)^{n/2}\sqrt{\operatorname{tr}_{a_t}((a_t')^2)\det(A_t)}\, dt \\ &\leq \int_0^1 \sqrt{\langle a_t', a_t'\rangle^0_{a_t}}\, dt = L^0(a_t) \leq \theta_x^g(g_0(x),g_1(x)) + \epsilon.\end{aligned}$$

Since x and ϵ were arbitrary, we have the desired result. ☐

A simple consequence is that a Θ_M-convergent sequence that is multiplied with such a function ρ is still Θ_M-convergent.

Next, we need to (algebraically) extend the exponential mapping as given by the expression in Proposition 2.1 to a more general class of functions and basepoints. In the following, we will make the maximal such definition that still guarantees that the image of this extended "exponential mapping" has finite volume if the basepoint does.

Lemma 4.6. *For each $\tilde{g} \in \mathcal{M}_f$, define*

$$\mathcal{F}_{\tilde{g}} := \left\{ \zeta \in L^2(M,\tilde{g}) \;\middle|\; \zeta(x) \geq -\frac{4}{n} \text{ for all } x \in M \right\}.$$

Then there exists a map

$$\begin{aligned}\psi_{\tilde{g}} : \mathcal{F}_{\tilde{g}} &\to \mathcal{M}_f \\ \zeta &\mapsto \left(1 + \frac{n}{4}\zeta\right)^{4/n}\tilde{g}.\end{aligned} \tag{4.9}$$

(Note that $\psi_{\tilde{g}}(\zeta)$ is formally the same expression as $\exp_{\tilde{g}}(\zeta\tilde{g})$, if $\tilde{g} \in \mathcal{M}$ and $\zeta \in C^\infty(M)$ with $\zeta > -\frac{4}{n}$.)

Proof. We have to prove that if $\zeta \in \mathcal{F}_{\tilde{g}}$, then $\psi_{\tilde{g}}(\zeta) \in \mathcal{M}_f$. From (4.9) and the definition of $\mathcal{F}_{\tilde{g}}$, it is clear that $\psi_{\tilde{g}}(\zeta)$ is a semimetric, since the conformal factor in front of $\tilde{g}$ is nonnegative.

To see that $\psi_{\tilde{g}}(\zeta)$ has finite volume, note that

$$\mu_{\psi_{\tilde{g}}} = \left(1 + \frac{n}{4}\zeta\right)^2 \mu_{\tilde{g}} = \left(1 + \frac{n}{2}\zeta + \frac{n^2}{16}\zeta^2\right)\mu_{\tilde{g}}.$$

Thus, finite volume would follow if we could show that each summand in the parentheses on the right is in $L^1(M,\tilde{g})$. But the constant function $1 \in L^2(M,\tilde{g})$ by finite volume of $\mu_{\tilde{g}}$. Furthermore, $\zeta^2 \in L^1(M,\tilde{g})$ since $\zeta \in L^2(M,\tilde{g})$. Finally, as is well known (it is a simple consequence of Hölder's Inequality), finite volume of $(M,\tilde{g})$ implies that $L^2(M,\tilde{g}) \subset L^1(M,\tilde{g})$. Therefore, $\zeta \in L^1(M,\tilde{g})$, completing what was to be shown. ☐

We will retain the notation $\mathcal{F}_{\tilde{g}}$ and $\psi_{\tilde{g}}$ from the previous lemma for the remainder of the section.

The following two lemmas will allow us to control the distance between different conformal changes of a semimetric—one of the goals we outlined at the beginning of this subsection.

We need to first restrict to basepoint semimetrics that are bounded, and can then extend this to the general case.

Lemma 4.7. *Let $\tilde{g} \in \mathcal{M}_f$ be a bounded, measurable semimetric, and let $\kappa, \lambda \in \mathcal{F}_{\tilde{g}}$. Then*

$$d(\psi_{\tilde{g}}(\kappa), \psi_{\tilde{g}}(\lambda)) \leq \sqrt{n}\, \|\lambda - \kappa\|_{\tilde{g}}\,.$$

Proof. By the proof of [3, Thm. 5.14], we can find sequences $\{\kappa_k\}$ and $\{\lambda_k\}$ of smooth functions with the following properties:

- $\{\kappa_k\}$ and $\{\lambda_k\}$ converge in $L^2(M, \tilde{g})$ to κ and λ, respectively,
- $\kappa_k, \lambda_k > -\frac{4}{n}$ for all $k \in \mathbb{N}$, and
- we have
$$\lim_{k\to\infty} d(\psi_{\tilde{g}}(\kappa_k), \psi_{\tilde{g}}(\kappa)) = 0 = \lim_{k\to\infty} d(\psi_{\tilde{g}}(\lambda_k), \psi_{\tilde{g}}(\lambda)).$$

Furthermore, [3, Lemma 5.16] implies

$$d(\psi_{\tilde{g}}(\kappa_k), \psi_{\tilde{g}}(\lambda_k)) \leq \sqrt{n}\, \|\lambda_k - \kappa_k\|_{\tilde{g}}\,.$$

Thus, the triangle inequality gives

$$\begin{aligned} d(\psi_{\tilde{g}}(\kappa), \psi_{\tilde{g}}(\lambda)) &\leq \lim_{k\to\infty} [d(\psi_{\tilde{g}}(\kappa), \psi_{\tilde{g}}(\kappa_k)) + d(\psi_{\tilde{g}}(\kappa_k), \psi_{\tilde{g}}(\lambda_k)) + d(\psi_{\tilde{g}}(\lambda_k), \psi_{\tilde{g}}(\lambda))] \\ &\leq \lim_{k\to\infty} \sqrt{n}\, \|\lambda_k - \kappa_k\|_{\tilde{g}} = \sqrt{n}\, \|\lambda - \kappa\|_{\tilde{g}}\,. \end{aligned}$$

□

Lemma 4.8. *Let $\tilde{g} \in \mathcal{M}_f$ be any element, and let $\kappa, \lambda \in \mathcal{F}_{\tilde{g}}$. Then*

$$d(\psi_{\tilde{g}}(\kappa), \psi_{\tilde{g}}(\lambda)) \leq \sqrt{n}\, \|\lambda - \kappa\|_{\tilde{g}}\,.$$

Proof. Let ξ be a positive, measurable function such that $\tilde{g}_0 := \xi \tilde{g}$ is bounded. For the remainder of the proof, we abbreviate $\psi := \psi_{\tilde{g}}$ and $\psi_0 := \psi_{\tilde{g}_0}$. Define

$$\kappa_0 := \xi^{-n/4}\left(\kappa + \frac{4}{n} - \xi^{n/4}\right) \qquad \text{and} \qquad \lambda_0 := \xi^{-n/4}\left(\lambda + \frac{4}{n} - \xi^{n/4}\right).$$

Then a simple calculation shows that $\psi_0(\kappa_0) = \psi(\kappa)$ and $\psi(\lambda_0) = \psi(\lambda)$.

But on the other hand, since $\tilde{g}_0$ is bounded, we can apply the previous lemma to get

$$\begin{aligned} d(\psi_0(\kappa_0), \psi_0(\lambda_0)) &\leq \sqrt{n}\, \|\lambda_0 - \kappa_0\|_{\tilde{g}_0} \\ &= \sqrt{n}\left(\int_M \left(\xi^{-n/4}\left(\lambda + \frac{4}{n} - \xi^{n/4}\right) - \right.\right. \\ &\qquad\qquad \left.\left. \xi^{-n/4}\left(\kappa + \frac{4}{n} - \xi^{n/4}\right)\right)^2 d\mu_{\tilde{g}_0}\right)^{1/2} \\ &= \sqrt{n}\left(\int_M (\lambda - \kappa)^2\, \xi^{-n/2}\, d\mu_{\tilde{g}_0}\right)^{1/2} = \sqrt{n}\left(\int_M (\lambda - \kappa)^2\, d\mu_{\tilde{g}}\right)^{1/2} \\ &= \sqrt{n}\, \|\lambda - \kappa\|_{\tilde{g}}\,. \end{aligned}$$

This gives the desired result. □

The last technical result that we will need for the moment concerns the behavior of the norms of bounded functions with respect to a d-convergent sequence of semimetrics.

Lemma 4.9. *Let $g_k, g_0 \in \mathcal{M}_f$ with $g_k \xrightarrow{\Theta_M} g_0$, and let λ be a bounded, measurable function on M. Then $\lambda \in L^2(M, g_i)$ for all $i = 0, 1, 2, \ldots$, and $\|\lambda\|_{g_k} \to \|\lambda\|_{g_0}$.*

Proof. We note that for $i = 0, 1, 2, \ldots$,

$$\|\lambda\|_{g_i} = \left\| \lambda \sqrt{\left(\frac{\mu_{g_i}}{\mu_g} \right)} \right\|_g = \left(\int_M \lambda^2 \left(\frac{\mu_{g_i}}{\mu_g} \right) \, d\mu_g \right)^{1/2}.$$

But by Theorem 3.8,

$$\left(\frac{\mu_{g_k}}{\mu_g} \right) \xrightarrow{L^1(M,g)} \left(\frac{\mu_{g_0}}{\mu_g} \right).$$

The result then follows straightforwardly from this and from the boundedness of λ. □

With these results at hand, we can prove the equivalence of the topologies of d and Θ_M on $\mathcal{M}_f$.

Theorem 4.10. *Let $g_k, g_0 \in \mathcal{M}_f$. Then $g_k \xrightarrow{d} g_0$ if and only if $g_k \xrightarrow{\Theta_M} g_0$.*

Proof. As in Proposition 4.4, the only statement that needs proving is that $g_k \xrightarrow{\Theta_M} g_0$ implies $g_k \xrightarrow{d} g_0$.

So let $\epsilon > 0$ be given, and we wish to see that $d(g_k, g_0) < \epsilon$ for k large enough. The idea of the proof is to find a sequence $\{\sigma_l\}$ of measurable, positive functions with $\sigma_l \leq 1$ for all l, and with the property that $\sigma_l g_0$ is bounded. Then Lemma 4.5 and Proposition 4.4 apply to give that $\sigma_l g_k \xrightarrow{d} \sigma_l g_0$, as $k \to \infty$, for each l. Furthermore, if we can arrange that $\sigma_l g_0 \xrightarrow{d} g_0$, and we can say that $d(\sigma_l g_k, g_k)$ is close to $d(\sigma_l g_0, g_0)$ for large k, then we have estimated each term on the right-hand side of this double application of the triangle inequality:

$$d(g_k, g_0) \leq d(g_k, \sigma_l g_k) + d(\sigma_l g_k, \sigma_l g_0) + d(\sigma_l g_0, g_0).$$

So let us get down to the details of this argument.

We first choose a measurable, positive function ξ on M such that $\xi \leq 1$ and ξg_0 is bounded. Set $\rho := \xi^{-1}$ and $g_i^0 := \xi g_i$ for $i = 0, 1, 2, \ldots$, so that $g_i = \rho g_i^0$. A simple estimate using the finite volume of g_0 shows that $\rho \in L^{n/2}(M, g_0^0)$.

For each $i = 0, 1, 2, \ldots$, let's abbreviate $\psi_i := \psi_{g_i^0}$. Then, we set

$$\lambda := \frac{4}{n} \left(\rho^{n/4} - 1 \right). \tag{4.10}$$

Clearly $\psi_i(\lambda) = \rho g_i^0 = g_i$ for $i = 0, 1, 2, \ldots$. Moreover, we claim that $\lambda \in L^2(M, g_0^0)$ and hence we can find a sequence $\{\lambda_l\}$ of bounded, measurable functions on M that converge in $L^2(M, g_0^0)$ to λ as $l \to \infty$. That $\lambda \in L^2(M, g_0^0)$ follows from two facts. First, $\rho \in L^{n/2}(M, g_0^0)$, implying that $\rho^{n/4} \in L^2(M, g_0^0)$. Second, finite volume of g_0^0 implies that the constant function $1 \in L^2(M, g_0^0)$ as well.

Since $\rho > 0$, we have $\lambda > -\frac{4}{n}$. Thus, we can choose our λ_l such that $\lambda_l > -\frac{4}{n}$. We then have, by Lemma 4.8, that

$$d(\psi_0(\lambda_l), g_0) = d(\psi_0(\lambda_l), \psi_0(\lambda)) \leq \sqrt{n} \, \|\lambda - \lambda_l\|_{g_0^0}, \tag{4.11}$$

and so by our choice of λ_l,

$$\lim_{l \to \infty} d(\psi_0(\lambda_l), g_0) \leq \lim_{l \to \infty} \sqrt{n} \, \|\lambda - \lambda_l\|_{g_0^0} = 0. \tag{4.12}$$

It is possible to choose each λ_l such that $\lambda_l(x) \leq \lambda(x)$ for all $x \in M$. Define ρ_l to be the function such that $\psi_0(\lambda_l) = \rho_l g_0^0$, and set $\sigma_l := \rho_l \xi$. Thus,

$$\rho_l = \left(1 + \frac{n}{4}\lambda_l\right)^{4/n} \leq \left(1 + \frac{n}{4}\lambda\right)^{4/n} = \rho,$$

and since $g_i^0 = \xi g_i$, we have $\rho_l g_i^0 = \sigma_l g_i$ for $i = 0, 1, 2, \ldots$. But since $0 < \rho_l \leq \rho$ and $0 < \xi = \rho^{-1}$, we see that $0 < \sigma_l \leq 1$ for all $l \in \mathbb{N}$. Thus, by Lemma 4.5, $\psi_k(\lambda_l) = \sigma_l g_k \xrightarrow{\Theta_M} \sigma_l g_0 = \psi_0(\lambda_l)$, as $k \to \infty$, for each $l \in \mathbb{N}$. Additionally, since $\sigma_l g_0 = \rho_l g_0^0$ is bounded (because each λ_l is bounded by assumption), Proposition 4.4 gives that

$$\lim_{k\to\infty} d(\psi_k(\lambda_l), \psi_0(\lambda_l)) = 0 \tag{4.13}$$

for each $l \in \mathbb{N}$.

Now, as above, Lemma 4.8 gives

$$d(\psi_k(\lambda_l), g_k) = d(\psi_k(\lambda_l), \psi_k(\lambda)) \leq \sqrt{n}\, \|\lambda - \lambda_l\|_{g_k^0}. \tag{4.14}$$

Since $g_k \xrightarrow{\Theta_M} g_0$ and $\xi \leq 1$, Lemma 4.5 implies that $g_k^0 = \xi g_k \xrightarrow{\Theta_M} \xi g_0 = g_0^0$. Thus, Lemma 4.9 yields

$$\lim_{k\to\infty} d(\psi_k(\lambda_l), g_k) \leq \lim_{k\to\infty} \sqrt{n}\, \|\lambda - \lambda_l\|_{g_k^0} = \sqrt{n}\, \|\lambda - \lambda_l\|_{g_0^0}. \tag{4.15}$$

(We have used (4.14) to obtain the inequality and Lemma 4.9 to obtain the equality.)

This gives us all the pieces we need to estimate $d(g_k, g_0)$ using the triangle inequality. By (4.15), we can choose k large enough that

$$d(g_k, \psi_k(\lambda_l)) \leq \sqrt{n}\, \|\lambda - \lambda_l\|_{g_0^0} + \frac{\epsilon}{4}.$$

By (4.13), we can also choose k large enough that

$$d(\psi_k(\lambda_l), \psi_0(\lambda_l)) < \frac{\epsilon}{4}.$$

Finally, using (4.12), we can choose l large enough that

$$\sqrt{n}\, \|\lambda - \lambda_l\|_{g_0^0} < \frac{\epsilon}{4} \qquad \text{and} \qquad d(\psi_0(\lambda_l), g_0) < \frac{\epsilon}{4}.$$

Putting together these four inequalities completes the proof of the theorem. □

In view of Theorem 3.8, we get the following immediate corollary, which is perhaps of independent interest. It will also be useful in the next subsection.

Corollary 4.11. *Let $g_k, g_0 \in \mathcal{M}_f$, and let $g_k \xrightarrow{d} g_0$. Then*

$$\left(\frac{\mu_{g_k}}{\mu_g}\right) \xrightarrow{L^1(M,g)} \left(\frac{\mu_{g_0}}{\mu_g}\right).$$

Equivalently, μ_{g_k} converges uniformly to μ_{g_0}.

At this point, we wish to move on to studying the relation between the completions of $(\mathcal{M}, d)$ and $(\mathcal{M}, \Theta_M)$, which amounts to investigating which sequences are Cauchy with respect these metrics, and when two Cauchy sequences are equivalent. But because pointwise, $\overline{(\mathcal{M}, d)}$ and $\overline{(\mathcal{M}, \Theta_M)}$ are in bijection with one another (through their respective bijections with $\widehat{\mathcal{M}_f}$), we can avoid direct considerations of these issues. In fact, there is little argumentation yet remaining.

Theorem 4.12. *A sequence in $\{g_k\} \subset \mathcal{M}$ is d-Cauchy if and only if it is Θ_M-Cauchy.*

Proof. That a d-Cauchy sequence is Θ_M-Cauchy is immediate from Proposition 3.3. So we turn to the converse statement.

Let $\{g_k\}$ be Θ_M-Cauchy, and let $g_0 \in \mathcal{M}_f$ be a representative of its Θ_M-limit in $\widehat{\mathcal{M}_f}$ as guaranteed by Theorem 3.9. Then $g_k \xrightarrow{\Theta_M} g_0$, implying that $g_k \xrightarrow{d} g_0$. But as a d-convergent sequence, $\{g_k\}$ is necessarily d-Cauchy, as was to be proved. □

Theorem 4.13. *There exist natural homeomorphisms $\overline{(\mathcal{M}, d)} \cong \overline{(\mathcal{M}, \Theta_M)} \cong \widehat{\mathcal{M}_f}$.*

Proof. We know that a d-Cauchy sequence is Θ_M-Cauchy and vice versa, as well as that both completions can be identified with $\widehat{\mathcal{M}_f}$. We must still see that two Cauchy sequences $\{g_k^0\}$ and $\{g_k^1\}$ are d-equivalent (i.e., $\lim_{k\to\infty} d(g_k^0, g_k^1) = 0$) if and only if they are Θ_M-equivalent (i.e., $\lim_{k\to\infty} \Theta_M(g_k^0, g_k^1) = 0$). But assume $\lim_{k\to\infty} d(g_k^0, g_k^1) = 0$. Since $\widehat{\mathcal{M}_f}$ is complete with respect to d and $\{g_k^0\}$ and $\{g_k^1\}$ are d-Cauchy, they d-converge to some elements g_0 and g_1, respectively. By Theorem 4.10, the sequences Θ_M-converge to g_0 and g_1 as well. However, since $\lim_{k\to\infty} d(g_k^0, g_k^1) = 0$, we must have that $g_0 = g_1$, from which we conclude that $\lim_{k\to\infty} \Theta_M(g_k^0, g_k^1) = 0$. The converse statement is proved in exactly the same way. □

4.3. **Another characterization of convergence in $\mathcal{M}$.** The last major result of the paper is another characterization of convergence in $\mathcal{M}$ that does not require reference to either d or Θ_M. We will state it after a brief lemma. After showing the convergence result using Theorem 4.10, we use it to prove the discontinuity of various geometric quantities on $\mathcal{M}$.

Lemma 4.14. *Let $\{g_k\} \subset \mathcal{M}_f$ Θ_M-converge to $g_0 \in \mathcal{M}_f$. Then for each representative $g_0 \in [g_0]$, $\{g_k\}$ $|\cdot|_{g(x)}$-converges to g_0 in measure on $M \setminus D_{\{g_k\}} = M \setminus X_{g_0}$.*

Proof. Let $\delta, \epsilon > 0$ be given; we must find $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies that if

$$Y_k^\delta := \left\{ x \in M \;\middle|\; |g_0(x) - g_k(x)|_{g(x)} \geq \delta \right\},$$

then $\mathrm{Vol}(Y_k^\delta \setminus X_{g_0}, g) < \epsilon$.

For $\zeta, \tau > 0$, we define (as in Lemma 4.2)

$$\mathcal{M}_x^{\zeta,\tau} := \{a \in \mathcal{M}_x \mid \det A \geq \zeta,\ |a_{ij}| \leq \tau \text{ for all } 1 \leq i, j \leq n\}.$$

Choose ζ small enough and τ large enough that if

$$E^{\zeta,\tau} := \left\{ x \in M \;\middle|\; g_0(x) \in \mathcal{M}_x^{\zeta,\tau} \right\},$$

then $\mathrm{Vol}(M \setminus (X_{g_0} \cup E^{\zeta,\tau}), g) < \epsilon/2$.

Furthermore, let η be defined as in Lemma 4.2. By Theorem 3.7, $\{g_k\}$ θ_x^g-converges to g_0 in measure, so we can find $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies that for

$$Z_k^\delta := \left\{ x \in E^{\zeta,\tau} \;\middle|\; \theta_x^g(g_k(x), g_0(x) \geq \eta(\delta) \right\},$$

$\mathrm{Vol}(Z_k^\delta, g) < \epsilon/2$. On the other hand, we have by definition that

$$|g_0(x) - g_k(x)|_{g(x)} < \delta$$

for $x \in E^{\zeta,\tau} \setminus Z_k^\delta$, implying that $Y_k^\delta \cap E^{\zeta,\tau} \subseteq Z_k^\delta$, so

$$\mathrm{Vol}(Y_k^\delta \setminus X_{g_0}, g) \leq \mathrm{Vol}(M \setminus (X_{g_0} \cup E^{\zeta,\tau}), g) + \mathrm{Vol}(Z_k^\delta, g) < \epsilon,$$

as was to be shown. □

Theorem 4.15. *Say $\{g_k\} \in \mathcal{M}_f$ and $g_0 \in \mathcal{M}$. Then $g_k \xrightarrow{d} g_0$ if and only if*

(1) *$g_k \to g_0$ in measure,*

and additionally one of the following conditions holds:

2a. *μ_k converges to μ uniformly; or*

2b.
$$\left(\frac{\mu_{g_k}}{\mu_g}\right) \xrightarrow{L^1(M,g)} \left(\frac{\mu_{g_0}}{\mu_g}\right).$$

Remark 4.16. Recall that by our terminology (cf. Definition 2.21), condition 1 means that g_k $|\cdot|_{g(x)}$-converges to g_0 in measure.

Proof of Theorem 4.15. The equivalence of conditions 2a and 2b is given by Lemma 2.25, so we will simply work with condition 2a.

First, suppose that $g_k \xrightarrow{d} g_0$. Then conditions 2a and 2b are immediately implied by Corollary 4.11. Condition 1 follows from applying Theorem 4.10 followed by Lemma 4.14. (Note that in this case, $D_{\{g_k\}} = X_{g_0} = \emptyset$, since $g_0 \in \mathcal{M}$.)

Conversely, let conditions 1 and 2a hold. By Theorem 4.10, it suffices to show that $g_k \xrightarrow{\Theta_M} g_0$.

Let $\epsilon > 0$ be given. By an argument exactly analogous to that in the proof of the last lemma, condition 1 implies that g_k θ^g_x-converges to g_0 in measure. Lemma 2.22 then allows us to take the measure it converges in to be μ_{g_0} instead of μ_g as usual. This means that if
$$E^\epsilon_k := \{x \in M \mid \theta^g_x(g_k(x), g_0(x)) \geq \epsilon\},$$
then for k large enough,
$$\mathrm{Vol}(E^\epsilon_k, g_0) < \epsilon. \tag{4.16}$$

For each $k \in \mathbb{N}$, define $g^0_k := \chi(M \setminus E^\epsilon_k) g_k + \chi(E^\epsilon_k) g_0$. Then $\theta^g_x(g^0_k(x), g_0(x)) = 0$ for $x \in E^\epsilon_k$ and $g^0_k(x) = g_k(x)$ for $x \notin E^\epsilon_k$, so
$$\Theta_M(g^0_k, g_0) = \int_{M \setminus E^\epsilon_k} \theta^g_x(g_k(x), g_0(x))\, d\mu_g < \mathrm{Vol}(M, g) \cdot \epsilon. \tag{4.17}$$

On the other hand, by Proposition 3.11, we have that
$$\Theta_M(g_k, g^0_k) \leq C'(n)(\mathrm{Vol}(E^\epsilon_k, g_k) + \mathrm{Vol}(E^\epsilon_k, g_0)),$$
since g^0_k and g_k differ only on E^ϵ_k, where $g^0_k = g_0$. By condition 2a and (4.16), the above thus implies that for k large enough, $\Theta_M(g_k, g^0_k) \leq 3C'(n) \cdot \epsilon$. Thus, by (4.17) and the triangle inequality, we conclude
$$\Theta_M(g_k, g_0) \leq (\mathrm{Vol}(M, g) + 3C'(n)) \cdot \epsilon.$$
Since ϵ was arbitrary, we have shown the desired result. ∎

This characterization of convergence is the most useful in practice, essentially since it is so weak and easy to check. Unfortunately, this weakness is exactly what causes so many problems when attempting to study the pseudometric induced on $\mathcal{M}/\mathcal{D}$ by d (see the Introduction). The first question that one would ask in this context is whether d induces a metric space structure on $\mathcal{M}/\mathcal{D}$. If this is not true, then we can find metrics g_0 and g_1 in separate diffeomorphism orbits and a sequence $\{\varphi_k\} \subset \mathcal{D}$ such that
$$\lim_{k\to\infty} d(\varphi^*_k g_0, g_1) = 0. \tag{4.18}$$

It seems very difficult to obtain an obstruction to this situation given the characterization of convergence in Theorem 4.15. (The only one that one we know of can be immediately read off—if $\mathrm{Vol}(M, g_0) \neq \mathrm{Vol}(M, g_1)$, then (4.18) cannot occur thanks to Lemma 2.17.) In particular, if some diffeomorphism-invariant geometric data were continuous with respect to d, then d would separate $\mathcal{D}$-orbits with varying data. However, as the following collection of examples shows, the most obvious geometric data is, in fact, discontinuous in a strong way.

To be precise, let J define some geometric data, that is, a map $\mathcal{M} \times M \to Y$, where Y is some metric space with metric δ. (For example, if J is scalar curvature, then J maps into $\mathbb{R}$.) We say that J is *continuous in measure* at $g_0 \in \mathcal{M}$ if for all $g_k \xrightarrow{d} g_0$ and all $\epsilon > 0$, we have

$$\lim_{k\to\infty} \mu_g \left(\{p \in M \mid \delta(J(g_k, p), J(g_0, p)) \geq \epsilon\}\right) = 0.$$

Example 4.17. Let $M = T^2$, the two-dimensional torus, with its standard chart. (We take this to be the rectangle $[-1, 1] \times [-1, 1]$, in the xy-plane $\mathbb{R}^2$, with opposite edges identified.) Let g_0 be the flat metric induced on T^2 via restriction of the Euclidean metric on $\mathbb{R}^2$ to this chart. Then the following sequences $\{g_k\}$ show that basic geometric data are discontinuous in measure at g_0.

Curvature. If J is any type of curvature, then it is clear from Theorem 4.15 that J is discontinuous in measure on $\mathcal{M}$.

Distance function. Let J be the distance function of the metric, i.e., $J(\tilde{g}, p) = d(p, \cdot) \in C^0(M)$. For each $s \in (0, \frac{1}{2}]$, let f_s be a smooth function defined on $[-1, 1]$ such that $1 \leq f_s(t) \leq s^{-4}$ for all $t \in [-1, 1]$, $f_s(s) = f_s(-s) = s^{-4}$, and $f_s(2s) = f_s(-2s) = 1$. Finally, let g_k be the sequence given by

$$g_k(x,y) := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & x \in [-1, -2/k] \cup [2/k, 1], \\ \begin{pmatrix} f_{1/k}(x) & 0 \\ 0 & f_{1/k}^{-1}(x) \end{pmatrix}, & x \in (-2/k, -1/k) \cup (1/k, 2/k), \\ \begin{pmatrix} k^4 & 0 \\ 0 & k^{-4} \end{pmatrix}, & x \in [-1/k, 1/k]. \end{cases}$$

It is not hard to see from Theorem 4.15 that $g_k \xrightarrow{d} g_0$. Furthermore, since $g_k(x, y)$ is constant in y, we see that the geodesics connecting points with equal y-coordinates are horizontal lines. On the other hand, one also easily computes that the length, with respect to g_k, of horizontal lines passing all the way through the cylindrical region $\{(x, y) \mid x \in [-1/k, 1/k]\}$ is unbounded as $k \to \infty$. Thus the distance function is discontinuous in measure at g_0—geometrically, the torus converges a cylinder with two infinitely long cusps as ends. (This is provided the functions f_s are chosen "well". The convergence can be taken to be, e.g., pointed Gromov–Hausdorff convergence. See [9, §3.B].) Note that more specifically, we have shown that the distance function is not "upper semicontinuous in measure" on $\mathcal{M}$.

Diameter. Let $J(\tilde{g}, p)$ be the diameter, i.e., $J(\tilde{g}, p) = \mathrm{diam}(M, \tilde{g})$ independently of p. Clearly, the above example shows that this is also discontinuous in measure at g_0.

Injectivity radius. Let $J(\tilde{g}, p) = \mathrm{inj}_{\tilde{g}}(p)$. Fix $k \in \mathbb{N}$, and define a region $E_k \subset T^2$ by

$$E_k := \{(x, y) \mid x \in [-3/4, 3/4],\ y \in [-1/k, 1/k]\}.$$

Note that $\mathrm{Vol}(E_k, g_0) = 3/k$. Let U_k be an open domain with $E_k \subset U_k$ and satisfying $\mathrm{Vol}(U_k, g_0) \leq 4/k$. Now, choose any metric $g_k \in \mathcal{M}$ such that $g_k(p) = g_0$ for $p \notin U_k$, and for $p \in E_k$,

$$g_k(p) = \begin{pmatrix} k^{-1} & 0 \\ 0 & k \end{pmatrix}.$$

Furthermore, we assume that g_k is chosen such that $(g_k)_{11}(p) \leq 1$ and $\mu_{g_k}(p) = \mu_{g_0}(p)$ for all $p \in U_k$. Then one can compute that, with respect to g_k, the length of the closed curve γ given by $\gamma(t) = (t, 0)$, for $t \in [-1, 1]$, satisfies

$$\limsup_{k \to \infty} L_{g_k}(\gamma) \leq \frac{1}{2}.$$

But from this, it is not hard to infer that $\limsup_{k\to\infty} \mathrm{inj}_{g_k}(p) < 1$ for p on a set of positive μ_g-measure. Since it is also clear by Theorem 4.15 that $g_k \xrightarrow{d} g_0$, this shows that the injectivity radius is discontinuous in measure at g_0. Note that this example also shows that the distance function is not even "lower semicontinuous in measure".

We remark that these examples do not give the situation of (4.18), since the metrics g_k in each are clearly mutually non-isometric. In fact, to date we do not have a single example where we can compute the d-distance between two elements of $\mathcal{M}/\mathcal{D}$ with equal total volumes, or even estimate this distance away from zero.

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